ECE 463/663 - Homework #9

Calculus of Variations. Ricatti Equation. LQG Control. Due Monday, April 7th

Soap Film

1) Calculate the shape of a soap film connecting two rings around the X axis:

- Y(0) = 8
- Y(5) = 7

Soap films minimize the suruface area of the soap film. This relates to the funcitonal

$$F = y\sqrt{1+\dot{y}^2}$$

From the lecture notes, the solution is of the form

$$y = a \cdot \cosh\left(\frac{x-b}{a}\right)$$

Plugging in the two endpoints, you get two equations and two unknowns

$$8 = a \cdot \cosh\left(\frac{0-b}{a}\right)$$

$$7 = a \cdot \cosh\left(\frac{5-b}{a}\right)$$

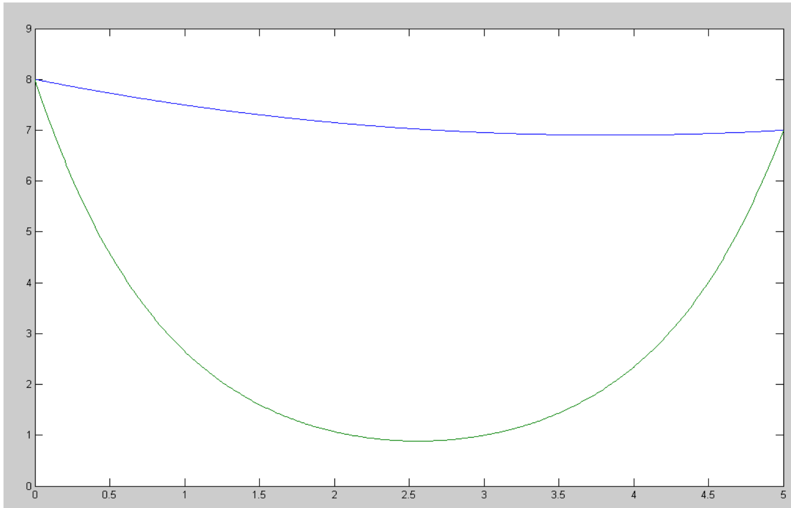
Set up a cost function in Matlab

```
function [J] = Soap(z)
   a = z(1);
   b = z(2);
   e1 = a*cosh((0-b)/a) - 8;
   e2 = a*cosh((5-b)/a) - 7;
   J = e1^2 + e2^2;
   end
```

Solving using *fminsearch()* and Matlab results in two solutions:

```
>> [Z,e] = fminsearch('Soap',[1,2])
Z =    0.8852    2.5595
e =  3.3498e-008

>> [Z,e] = fminsearch('Soap',[5,6])
Z =    6.9027    3.8423
e =  2.9778e-010
```

- a = 6.9027, b = 3.8424
- a = 0.8852 b = 2.5595



0
1
2
3
4
5
6
7
8
9
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5

2) Calculate the shape of a soap film connecting two rings around the X axis:

- Y(0) = 8
- Y(3) = free

The shape of the soap film is

$$y = a \cdot \cosh\left(\frac{x-b}{a}\right)$$

The left endpoint gives one constraint

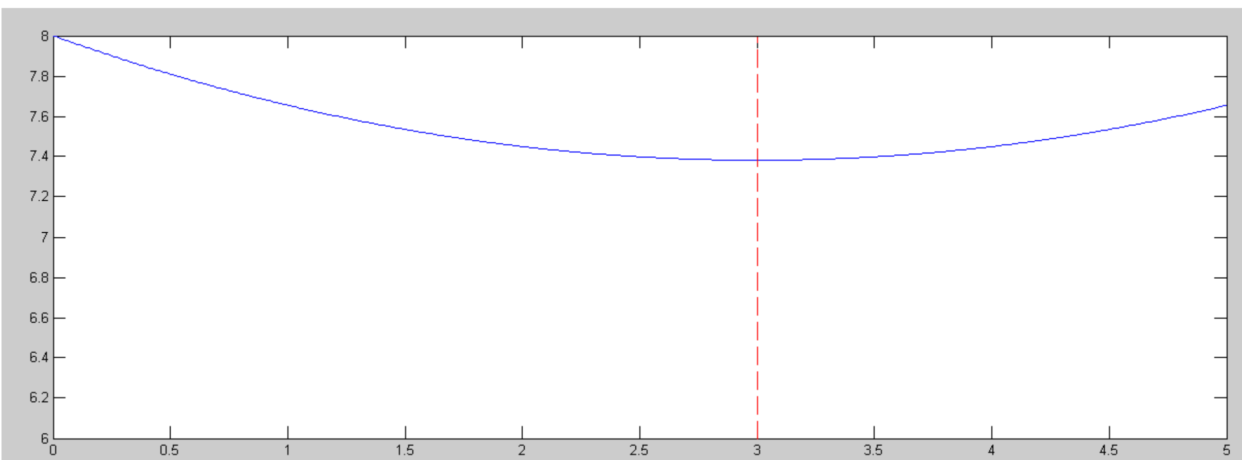
$$8 = a \cdot \cosh\left(\frac{0-b}{a}\right)$$

The right endpoint gives the constraint

$$\dot{y}(x=3) = 0 = \sinh\left(\frac{3-b}{a}\right)$$

This has the solution

- a = 7.3820
- b = 3



Hanging Chain

3) Calculate the shape of a hanging chain subject to the following constraints

- Length of chain = 15 meters
- Left Endpoint: (0,8)
- Right Endpoint: (10,7)

Hanging chains minimize the potential energy subject to a constraint that the length is 15. The functional to minimize is:

$$F = y\sqrt{1+\dot{y}^1} + M\sqrt{1+\dot{y}^2}$$

From the lecture notes, the solution is of the form

$$y = a \cdot \cosh\left(\frac{x-b}{a}\right) - M$$

$$L = \left(a \cdot \sinh\left(\frac{x-b}{m}\right)\right)_0^{10} = 15$$

Set up a cost function in Matlab

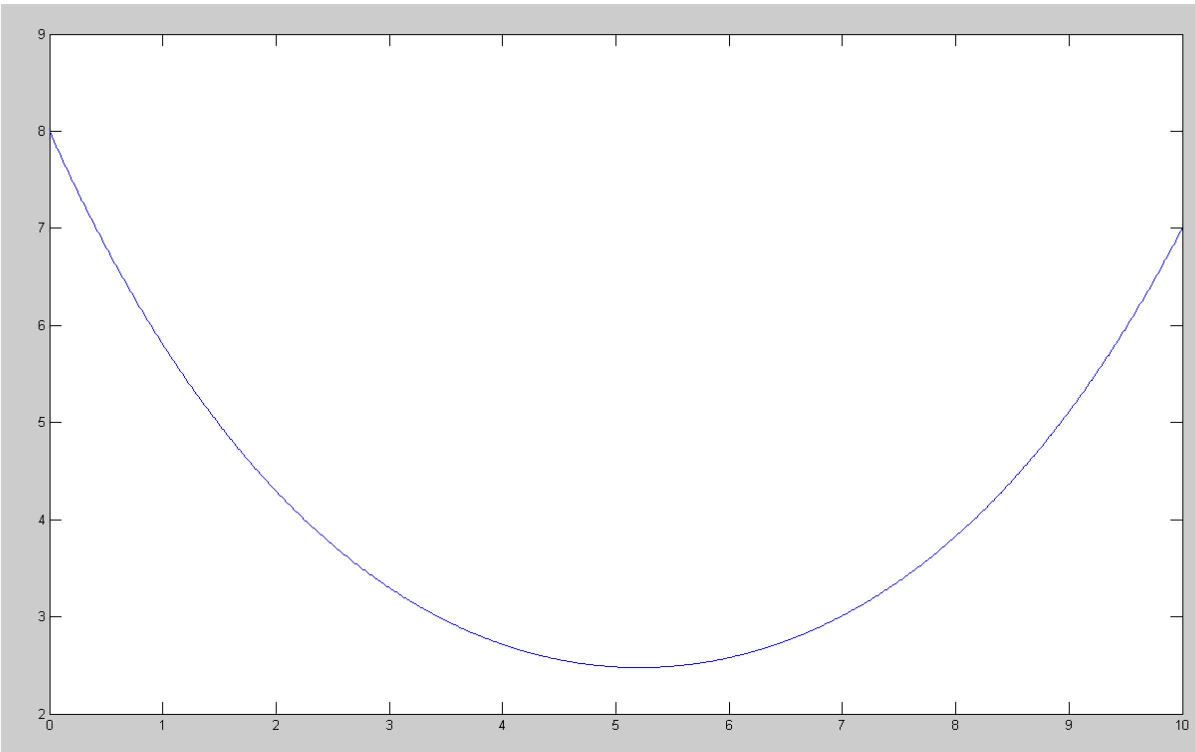
```
function J = chain(z)
a = z(1);
b = z(2);
M = z(3);

Length = 15;
x1 = 0;
y1 = 8;

x2 = 10;
y2 = 7;

e1 = a*cosh((x1-b)/a) - M - y1;
e2 = a*cosh((x2-b)/a) - M - y2;
e3 = a*sinh((x2-b)/a) - a*sinh((x1-b)/a) - Length;

J = e1^2 + e2^2 + e3^2;

end
```

Solve using *fminsearch()*

```
>> [Z,e] = fminsearch('chain',10*rand(1,3))

         a          b         M
Z =    3.0915    5.2064    0.6147

e =  1.1351e-008
```

This looks like the following:

```
>> a = Z(1);
>> b = Z(2);
>> M = Z(3);
>> x = [0:0.01:10]';
>> y = a*cosh((x-b)/a) - M;
>> plot(x,y)
```



Ricatti Equation

4) Find the function, x(t), which minimizes the following funcional

$$J = \int_0^{10} (x^2 + 10\dot{x}^2)dt$$

x(0) = 8

x(10) = 7

The functional is

$$F = x^2 + 10\dot{x}^2$$

The solution must satisfy the Euler LaGrange equation

$$F_x - \frac{d}{dt}(F_{\dot{x}}) = 0$$

$$2x - \frac{d}{dt}(20\dot{x}) = 0$$

$$2x - 20\ddot{x} = 0$$

Using the LaPlace operator

$$2(1 - 10s^2)X = 0$$

Either

- x = 0 (the trivial solution), or
- s = {+0.3162, -0.3162}

Going with the latter solution, x(t) is of the form

$$x(t) = ae^{0.3162t} + be^{-0.3162t}$$

Plugging in the endpoint constraints

$$x(0) = 8 = a + b$$

$$x(10) = 7 = 23.6243a + 0.0423b$$

Solving

$$a = 0.2825$$

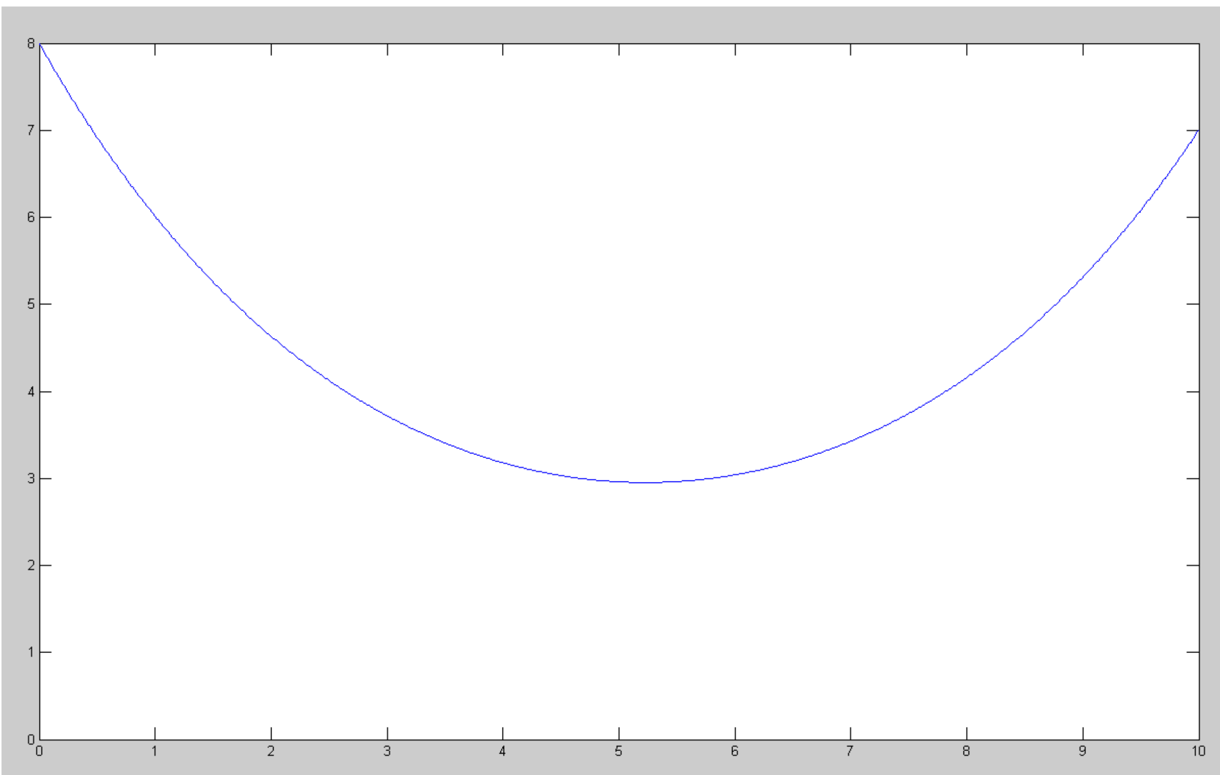
$$b = 7.7175$$

and

$$x(t) = 0.2825e^{0.3162t} + 7.7175e^{-0.3162t}$$

Plotting this in Matlab

```
>> B = [1,1 ; exp(10*sqrt(0.1)),exp(-10*sqrt(0.1))];
>> X = inv(B) * [8;7];

    0.2825
    7.7175

>> a = X(1)
a =    0.2825

>> b = X(2)
b =    7.7175

>> x = a*exp(sqrt(0.1)*t) + b*exp(-sqrt(0.1)*t);
>> plot(t,x)
>> ylim([0,8])
>>
```



5) Find the function, x(t), which minimizes the following funcional

$$J = \int_0^{10} (x^2 + 10u^2) dt$$

$$\dot{x} = -0.5x + u$$

$$x(0) = 8$$

$$x(10) = 7$$

The functional is

$$F = x^2 + 10u^2 + m(\dot{x} + 0.5x - u)$$

This gives three Euler LaGrange equations for {x, u, m}

x: $$F_x - \tfrac{d}{dt}(F_{\dot{x}}) = 0$$

$$(2x + 0.5m) - \tfrac{d}{dt}(m) = 0$$

$$(2x + 0.5m) - \dot{m} = 0$$

u: $$F_u - \tfrac{d}{dt}(F_{\dot{u}}) = 0$$

$$(20u - m) - \tfrac{d}{dt}(0) = 0$$

$$m = 20u$$

m: $$F_m - \tfrac{d}{dt}(F_{\dot{m}}) = 0$$

$$\dot{x} + 0.5x - u = 0$$

Solving, substitute for u

$$u = m/20$$

$$\dot{x} + 0.5x - m/20 = 0$$

$$m = 20\dot{x} + 10x$$

Substitute m into the first equation

$$2x + 0.5m - \dot{m} = 0$$

$$2x + 0.5(20\dot{x} + 10x) - (20\ddot{x} + 10\dot{x}) = 0$$

$$-20\ddot{x} + 7x = 0$$

Using LaPlace notation

$$(-20s^2+7)X=0$$

Either

- x(t) = 0 trivial solution,
- s = +0.5916, -0.5916

Going with the latter solution

$$x(t)=ae^{0.5916t}+be^{-0.5916t}$$

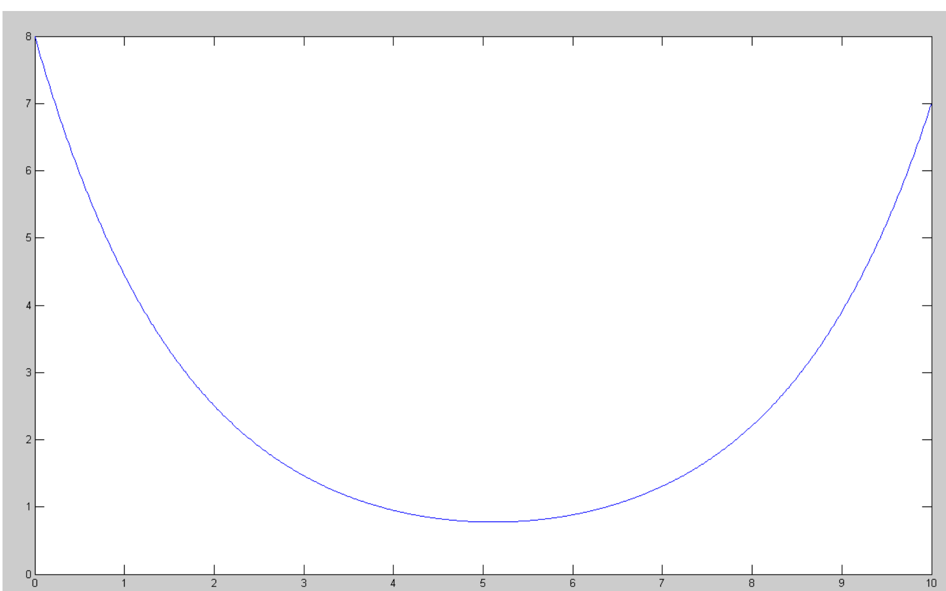
Plugging in the endpoint constraints

$$x(0)=8=a+b$$

$$x(10)=7=370.9546a+0.0027b$$

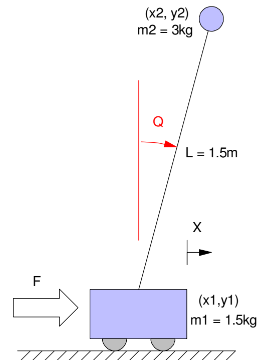
Solving

```
>> s = sqrt(7/20)
    0.5916

>> B = [1,1;exp(10*s),exp(-10*s)]

    1.0000    1.0000
  370.9546    0.0027

>> A = inv(B)*[8;7]

a     0.0188
b     7.9812

>> a = A(1);
>> b = A(2);
>> t = [0:0.01:10]';
>> x = a*exp(s*t) + b*exp(-s*t);
>> plot(t,x)
```



LQG Control for a Cart & Pendulum

6) Cart & Pendulum (HW #4 & HW#6):



$$s\begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -19.6 & 0 & 0 \\ 0 & 19.6 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0.667 \\ -0.444 \end{bmatrix} F$$

Design a full-state feedback control law of the form

$$F = U = K_r R - K_x X$$

for the cart and pendulum system from homework #4 using LQG control so that

- The DC gain is 1.00
- The 2% settling time is 6 seconds, and
- There is less than 10% overshoot for a step input.

Using trial and error, a good controller results from

```
A = [0,0,1,0;0,0,0,1;0,-19.6,0,0;0,19.6,0,0];
B = [0,0,0.6667,-0.4444]';
C = [1,0,0,0];
D = 0;

Qx = C'*C;
Qv = (C*A)'*(C*A);

Kx = lqr(A,B,Qx*20+Qv*0,1);
DC = -C*inv(A-B*Kx)*B;
Kr = 1/DC;
t = [0:0.01:8]';
G = ss(A-B*Kx, B*Kr, C, D);
y = step(G,t);
plot(t,y);
```

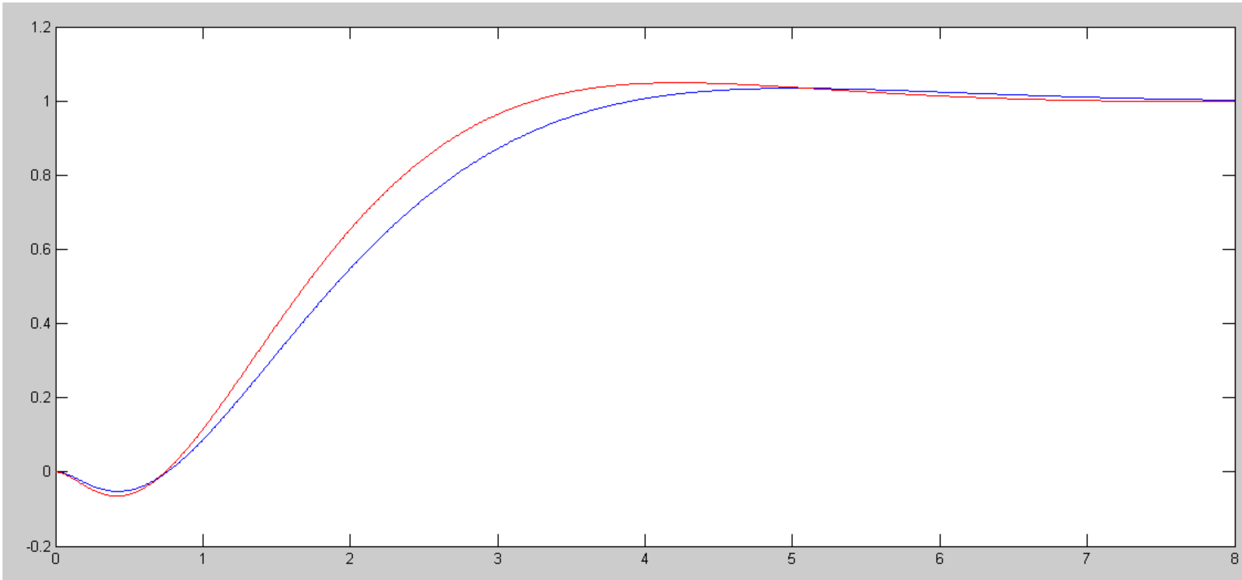
With this cost function, the 'optimal' control law is

```
Kx =   -4.4721 -126.6062   -8.6755   -36.2443

Kr =   -4.4721

>> eig(A-B*Kx)

  -4.4215 + 0.2253i
  -4.4215 - 0.2253i
  -0.7400 + 0.6682i
  -0.7400 - 0.6682i
```

Back in homework #6

```
Kx = -6.16    -133.78    -10.12   -39.093
```

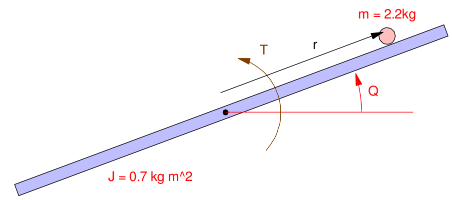
The gains are about the same



Step response from LQR (blue) and pole-placement (red)

7) Ball and Beam (HW #4 & HW#6):

$$s\begin{bmatrix} r \\ \theta \\ \dot{r} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -7 & 0 & 0 \\ -7/434 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} r \\ \theta \\ \dot{r} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.345 \end{bmatrix} T$$



Design a full-state feedback control law of the form

$$T = U = K_r R - K_x X$$

for the ball and beam system from homework #4 using LQG control so that

- The DC gain is 1.00
- The 2% settling time is 6 seconds, and
- There is less than 10% overshoot for a step input.

A reasonable response comes from

```
A = [0,0,1,0;0,0,0,1;0,-7,0,0;-7.434,0,0,0];
B = [0,0,0,0.3454]';
C = [1,0,0,0];
D = 0;

Kx = lqr(A,B,diag([1,0,70,580]), 1);
DC = -C*inv(A-B*Kx)*B;
Kr = 1/DC;
t = [0:0.01:8]';
G = ss(A-B*Kx, B*Kr, C, D);
y = step(G,t);
plot(t,y,t,0*t+1.1,'m--');
```

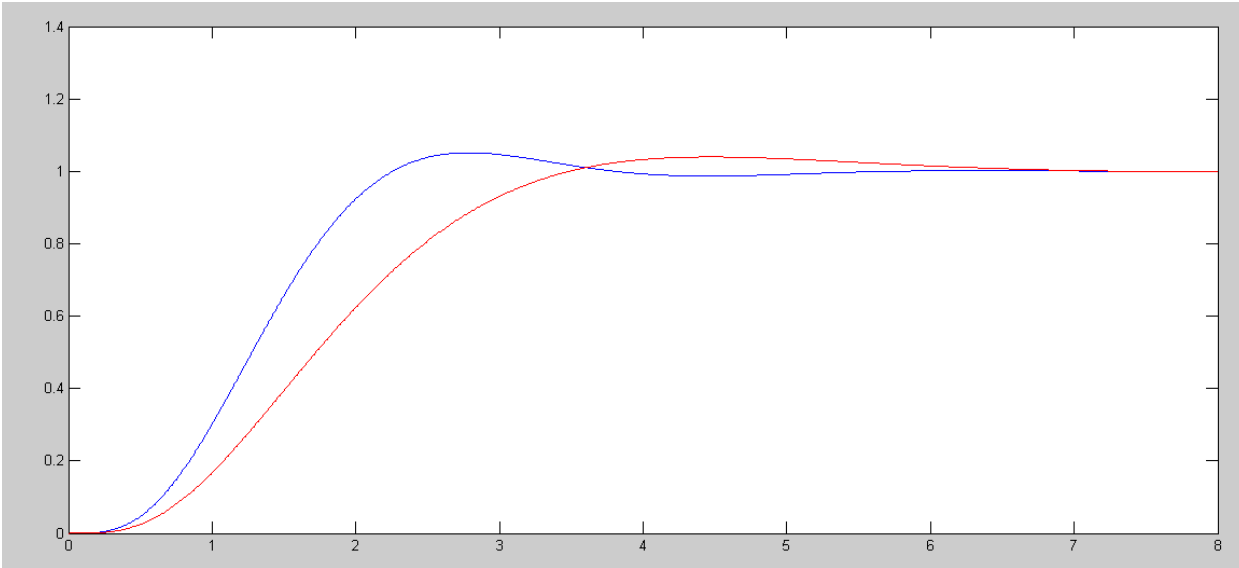
The optimal feedback gains with this cost funciton are:

```
Kx =   -43.0690   106.1646   -26.8989    34.5649

Kr =   -21.5461

>> eig(A-B*Kx)

  -8.4107
  -0.9736 + 1.7234i
  -0.9736 - 1.7234i
  -1.5809
```

Back in homework #6

```
Kx = -32.67     103.60     -18.25     -30.72
```

so the gains are pretty similar. The step response is maybe a little faster but similar.



Step Respons using LQR (blue) and Pole Placement (red)