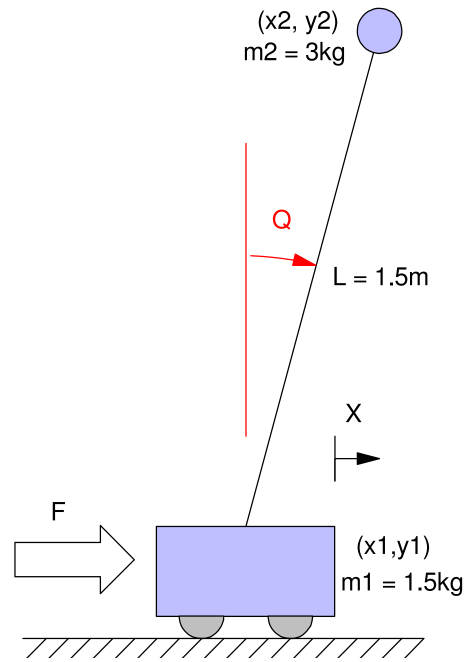
ECE 463/663 - Homework #4

LaGrangian Dynamics. Spring 2025

(30pt) Derive the dynamics for an inverted pendulum where

- m1 = 1.5kg (mass of cart)
- m2 = 3kg (mass of ball)
- L = 1.5m (length of arm)

Fine the linearized dynamics at x = 0, $\theta = 0$



Mass #1 (m1 = 1.5 kg)

$$x_1 = x \qquad y_1 = 0$$

$$\dot{x}_1 = \dot{x} \qquad \dot{y}_1 = 0$$

$$PE_1 = 0$$

$$KE_1 = \tfrac{1}{2} m v^2 = 0.75\dot{x}^2$$

Mass #2 (m2 = 3 kg, L = 1.5m)

$$x_2 = x + 1.5\sin\theta \qquad y_2 = 1.5\cos\theta$$

$$\dot{x}_2 = \dot{x} + 1.5\cos\theta\dot{\theta} \qquad \dot{y}_2 = -1.5\sin\theta\dot{\theta}$$

$$PE_2 = m_2 g y_2 = 3 \cdot g \cdot 1.5\cos\theta$$

$$KE_2 = \tfrac{1}{2} m_2 \left(\dot{x}_2^2 + \dot{y}_2^2 \right)$$

$$KE_2 = \tfrac{3}{2} \left(\left(\dot{x} + 1.5\cos\theta\dot{\theta} \right)^2 + \left(-1.5\sin\theta\dot{\theta} \right)^2 \right)$$

$$KE_2 = 1.5\dot{x}^2 + 3.375\dot{\theta}^2 \left(\cos^2\theta + \sin^2\theta \right) + 4.5\dot{x}\dot{\theta}\cos\theta$$

$$KE_2 = 1.5\dot{x}^2 + 3.375\dot{\theta}^2 + 4.5\dot{x}\dot{\theta}\cos\theta$$

The LaGrangian is then

$$L = KE - PE$$

$$L = (0.75\dot{x}^2) + \left(1.5\dot{x}^2 + 3.375\dot{\theta}^2 + 4.5\dot{x}\dot{\theta}\cos\theta \right) - (4.5g\cos\theta)$$

To find the dynamics, use the Euler LaGrange equation

$$L = \left(2.25\dot{x}^2 + 3.375\dot{\theta}^2 + 4.5\dot{x}\dot{\theta}\cos\theta\right) - (4.5g\cos\theta)$$

$$F = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \left(\frac{\partial L}{\partial x}\right)$$

$$F = \tfrac{d}{dt}\left(4.5\dot{x} + 4.5\cos\theta\dot{\theta}\right) - (0)$$

$$\boxed{F = 4.5\ddot{x} + 4.5\cos\theta\ddot{\theta} - 4.5\sin\theta\dot{\theta}^2}$$

$$L = \left(2.25\dot{x}^2 + 3.375\dot{\theta}^2 + 4.5\dot{x}\dot{\theta}\cos\theta\right) - (4.5g\cos\theta)$$

$$T = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \left(\frac{\partial L}{\partial \theta}\right)$$

$$T = \tfrac{d}{dt}\left(6.75\dot{\theta} + 4.5\cos\theta\dot{x}\right) - \left(-4.5\sin\theta\dot{x}\dot{\theta} + 4.5g\sin\theta\right)$$

$$T = 6.75\ddot{\theta} + 4.5\cos\theta\ddot{x} - 4.5\sin\theta\dot{x}\dot{\theta} + 4.5\sin\theta\dot{x}\dot{\theta} - 4.5g\sin\theta$$

$$\boxed{T = 6.75\ddot{\theta} + 4.5\cos\theta\ddot{x} - 4.5g\sin\theta}$$

So, the dynamics are

$$F = 4.5\ddot{x} + 4.5\cos\theta\ddot{\theta} - 4.5\sin\theta\dot{\theta}^2$$

$$T = 6.75\ddot{\theta} + 4.5\cos\theta\ddot{x} - 4.5g\sin\theta$$

In Matrix form

$$\begin{bmatrix} 4.5 & 4.5\cos\theta \\ 4.5\cos\theta & 6.75 \end{bmatrix}\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} F \\ T \end{bmatrix} + \begin{bmatrix} 4.5\sin\theta\dot{\theta}^2 \\ 4.5g\sin\theta \end{bmatrix}$$

Linearizing about zero with T = 0

$$\begin{bmatrix} 4.5 & 4.5 \\ 4.5 & 6.75 \end{bmatrix}\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}F + \begin{bmatrix} 0 \\ 4.5g\theta \end{bmatrix}$$

$$\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0.6667 \\ -0.4444 \end{bmatrix}F + \begin{bmatrix} -2g\theta \\ 2g\theta \end{bmatrix}$$

Putting this in state-space form with g = +9.8

$$s\begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -19.6 & 0 & 0 \\ 0 & 19.6 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0.6667 \\ -0.4444 \end{bmatrix}F$$

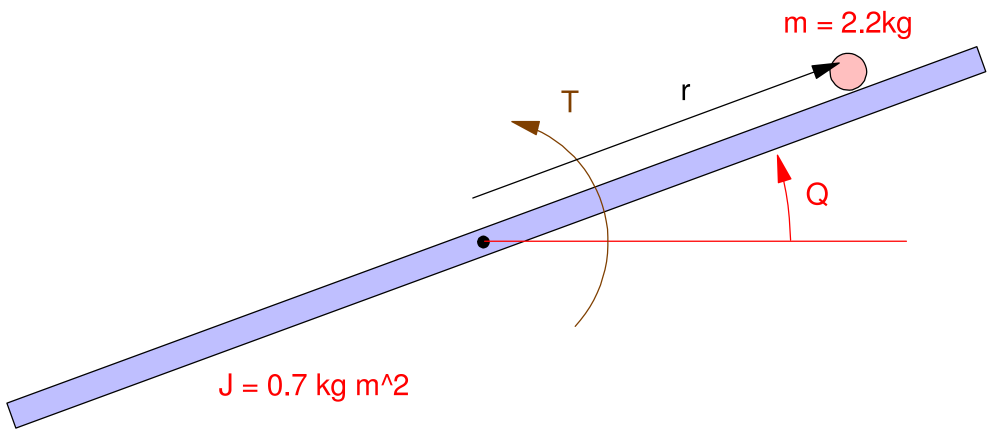
The poles are at

```
>>  A = [0,0,1,0;0,0,0,1;0,-19.6,0,0;0,19.6,0,0]


         0         0    1.0000         0
         0         0         0    1.0000
         0  -19.6000         0         0
         0   19.6000         0         0

>> eig(A)

         0
         0
    4.4272
   -4.4272
```

2) (30pt) Derive the dynamics for a ball and beam system where

- J = 0.7 kg m^2 (the inertia of the beam)
- m = 2.2 kg (the mass of the ball)

Find the linearized dynamics at r = 1.0m, θ = 0



Position of the ball:

$$x_1 = r\cos\theta \qquad y_1 = r\sin\theta$$

$$\dot{x}_1 = \dot{r}\cos\theta - r\sin\theta\dot{\theta} \qquad \dot{y}_1 = \dot{r}\sin\theta + r\cos\theta\dot{\theta}$$

The potential and kinetic energy. Assuming a solid sphere with radius 5mm (0.005m)

$$J = \tfrac{2}{5}mr^2 \qquad \textit{r = radius here}$$

$$x = r\theta \qquad \textit{x = displacement here}$$

$$\dot{x} = r\dot{\theta}$$

$$KE = \tfrac{1}{2}J\dot{\theta}^2 = \tfrac{1}{2}\left(\tfrac{2}{5}mr^2\right)\left(\tfrac{\dot{x}}{r}\right)^2 = \tfrac{1}{5}m\dot{x}^2 \qquad \textit{rotational KE for a solid sphere rolling}$$

This gives (using r for displacement along the beam for x)

$$PE = mgy_1 = mgr\sin\theta = 2.2gr\sin\theta$$

$$KE = \tfrac{1}{2}J\dot{\theta}^2 + \tfrac{1}{2}m\left(\dot{x}_1^2 + \dot{y}_1^2\right) + \tfrac{1}{5}m\dot{r}^2$$

$$KE = 0.35\dot{\theta}^2 + 1.1\left(\dot{x}_1^2 + \dot{y}_1^2\right) + 0.44\dot{r}^2$$

$$KE = 0.35\dot{\theta}^2 + 1.1\left(\left(\dot{r}\cos\theta - r\sin\theta\dot{\theta}\right)^2 + \left(\dot{r}\sin\theta + r\cos\theta\dot{\theta}\right)^2\right) + 0.44\dot{r}^2$$

$$KE = 0.35\dot{\theta}^2 + 1.1\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + 0.44\dot{r}^2$$

$$KE = (0.35 + 1.1r^2)\dot{\theta}^2 + 1.54\dot{r}^2$$

The LaGrangian is then

$$L = KE - PE$$

$$L = \left((0.35 + 1.1r^2)\dot{\theta}^2 + 1.54\dot{r}^2\right) - (2.2gr\sin\theta)$$

$$L = \left(0.35\dot{\theta}^2 + 1.1r^2\dot{\theta}^2 + 1.54\dot{r}^2\right) - (2.2gr\sin\theta)$$

Force on the Ball

$$L = \left(0.35\dot{\theta}^2 + 1.1r^2\dot{\theta}^2 + 1.54\dot{r}^2\right) - (2.2gr\sin\theta)$$

$$F = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \left(\frac{\partial L}{\partial r}\right)$$

$$F = \frac{d}{dt}(3.08\dot{r}) - \left(2.2r\dot{\theta}^2 - 2.2g\sin\theta\right)$$

$$\boxed{F = 3.08\ddot{r} - 2.2r\dot{\theta}^2 + 2.2g\sin\theta}$$

Torque on the Beam

$$L = \left(0.35\dot{\theta}^2 + 1.1r^2\dot{\theta}^2 + 1.54\dot{r}^2\right) - (2.2gr\sin\theta)$$

$$T = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \left(\frac{\partial L}{\partial \theta}\right)$$

$$T = \frac{d}{dt}\left(0.7\dot{\theta} + 2.2r^2\dot{\theta}\right) - (-2.2gr\cos\theta)$$

$$\boxed{T = 0.7\ddot{\theta} + 2.2r^2\ddot{\theta} + 4.4r\dot{r}\dot{\theta} + 2.2gr\cos\theta}$$

Putting it together

$$\begin{bmatrix} 3.08 & 0 \\ 0 & 0.7 + 2.2r^2 \end{bmatrix}\begin{bmatrix} \ddot{r} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 2.2r\dot{\theta}^2 - 2.2g\sin\theta \\ -4.4r\dot{r}\dot{\theta} - 2.2gr\cos\theta \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} T$$

Linearizing at r = 1.0m

$$\begin{bmatrix} 3.08 & 0 \\ 0 & 2.9 \end{bmatrix}\begin{bmatrix} \ddot{r} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} -2.2g\theta \\ -2.2gr \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} T$$

In State-Space form

$$s\begin{bmatrix} r \\ \theta \\ \dot{r} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -7 & 0 & 0 \\ -7.434 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} r \\ \theta \\ \dot{r} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.345 \end{bmatrix} T$$

The open-loop system is unstable

```
>> A = [0,0,1,0;0,0,0,1;0,-7,0,0;-7.434,0,0,0]

         0         0    1.0000         0
         0         0         0    1.0000
         0   -7.0000         0         0
   -7.4340         0         0         0

>> eig(A)

  -2.6858
  -0.0000 + 2.6858i
  -0.0000 - 2.6858i
   2.6858

>>
```