ECE 463/663 - Homework #3

Canonical Forms, Similarity Transforms, LaGrangian Dynamics, Block Diagrams
Due Monday, February 3rd

Canonical Forms

Problem 1-3) For the system

$$Y = \left(\frac{20\left(s^2+9\right)}{(s+2)(s+4)(s+6)} \right) U$$

1) Express this system in controller canonical form. (Give the A, B, C, D matrices)

Multiply out

$$Y = \left(\frac{20s^2+180}{s^3+12s^2+44s+48} \right) U$$

Almost by inspection, controller for is

$$sX = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -48 & -44 & -12 \end{bmatrix} X + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} U$$

$$Y = \begin{bmatrix} 180 & 0 & 20 \end{bmatrix} X + [0]U$$

Checking in Matlab (not required)

```
>> A = [0,1,0;0,0,1;-48,-44,-12]

     0     1     0
     0     0     1
   -48   -44   -12

>> B = [0;0;1]

     0
     0
     1

>> C = [180,0,20];
>> D = 0;
>> G = ss(A,B,C,D);
>> zpk(G)


   20 (s^2 + 9)
-----------------
(s+6) (s+4) (s+2)
```

2) Express this system in cascade form

$$Y = \left(\frac{20s^2+180}{(s+2)(s+4)(s+6)}\right) U$$

Rewrite this as

$$Y = \left(\left(\frac{a}{s+2}\right) + \left(\frac{b}{(s+2)(s+4)}\right) + \left(\frac{c}{(s+2)(s+4)(s+6)}\right)\right) U$$

Place over a common denominator

$$Y = \left(\left(\frac{a(s+4)(s+6)}{(s+2)(s+4)(s+6)}\right) + \left(\frac{b(s+6)}{(s+2)(s+4)(s+6)}\right) + \left(\frac{c}{(s+2)(s+4)(s+6)}\right)\right) U$$

The numberators must match up

$$20s^2 + 180 = a(s+4)(s+6) + b(s+6) + c$$

Solving

- a = 20
- b = -200
- c = 900

In state-space form then

$$sX = \begin{bmatrix} -2 & 0 & 0 \\ 1 & -4 & 0 \\ 0 & 1 & -6 \end{bmatrix} X + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} U$$

$$Y = \begin{bmatrix} 20 & -200 & 900 \end{bmatrix} X + [0] U$$

Checking in Matlab

```
>> A = [-2,0,0;1,-4,0;0,1,-6]

    -2     0     0
     1    -4     0
     0     1    -6

>> B = [1;0;0]

     1
     0
     0

>> C = [20,-200,900]

    20  -200   900

>> D = 0;
>> G = ss(A,B,C,D);
>> zpk(G)

  20 (s^2 + 9)
-----------------
(s+6) (s+4) (s+2)
```

3) Express this system in Jordan (diagonal) form

Do a partial fraction expansion

$$\left(\frac{20s^2+180}{(s+2)(s+4)(s+6)}\right) = \left(\frac{32.5}{s+2}\right) + \left(\frac{-125}{s+4}\right) + \left(\frac{112.5}{s+6}\right)$$

```
>> G = tf([20,0,180],[1,12,44,48])

      20 s^2 + 180
------------------------
s^3 + 12 s^2 + 44 s + 48

>> s = -2 + 1e-9;
>> a = evalfr(G,s) * (s+2)
a =   32.5000

>> s = -4 + 1e-9;
>> b = evalfr(G,s) * (s+4)
b = -125.0000

>> s = -6 + 1e-9;
>> c = evalfr(G,s) * (s+6)
c =  112.5000
```

In state-space

$$sX = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -6 \end{bmatrix} X + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} U$$

$$Y = \begin{bmatrix} 32.5 & -125 & 112.5 \end{bmatrix} X + [0] U$$

Checking in Matlab

```
>> A = [-2,0,0;0,-4,0;0,0,-6]

    -2     0     0
     0    -4     0
     0     0    -6

>> B = [1;1;1]

     1
     1
     1

>> C = [32.5, -125, 112.5]

C =

   32.5000 -125.0000  112.5000

>> D = 0;
>> G = ss(A,B,C,D);
>> zpk(G)

  20 (s^2 + 9)
-----------------
(s+6) (s+4) (s+2)

>>
```

4) Assume a system's dynamics are

$$\begin{bmatrix} sV_1 \\ sV_2 \\ sV_3 \\ sV_4 \end{bmatrix} = \begin{bmatrix} -3 & 1 & 0 & 0 \\ 1 & -3 & 1 & 0 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} V_0$$

$$Y = V_4$$

Express these dynamic with the change in variable

$$\begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \\ Z_4 \end{bmatrix} = \begin{bmatrix} V_1 - V_2 \\ V_2 - V_3 \\ V_3 - V_4 \\ V_4 \end{bmatrix}$$

```
>> A = [-3,1,0,0;1,-3,1,0;0,1,-3,1;0,0,1,-2]

    -3     1     0     0
     1    -3     1     0
     0     1    -3     1
     0     0     1    -2

>> B = [1;0;0;0];
>> C = [0,0,0,1];
>> D = 0;
>> Ti = [1,-1,0,0;0,1,-1,0;0,0,1,-1;0,0,0,1]

     1    -1     0     0
     0     1    -1     0
     0     0     1    -1
     0     0     0     1

>> T = inv(Ti)

     1     1     1     1
     0     1     1     1
     0     0     1     1
     0     0     0     1

>> Az = inv(T)*A*T

    -4     0    -1    -1
     1    -3     1     0
     0     1    -3     0
     0     0     1    -1

>> Bz = inv(T)*B

     1
     0
     0
     0

>> Cz = C*T

     0     0     0     1

>> Dz = D
```

```
Dz =

     0

>>
```

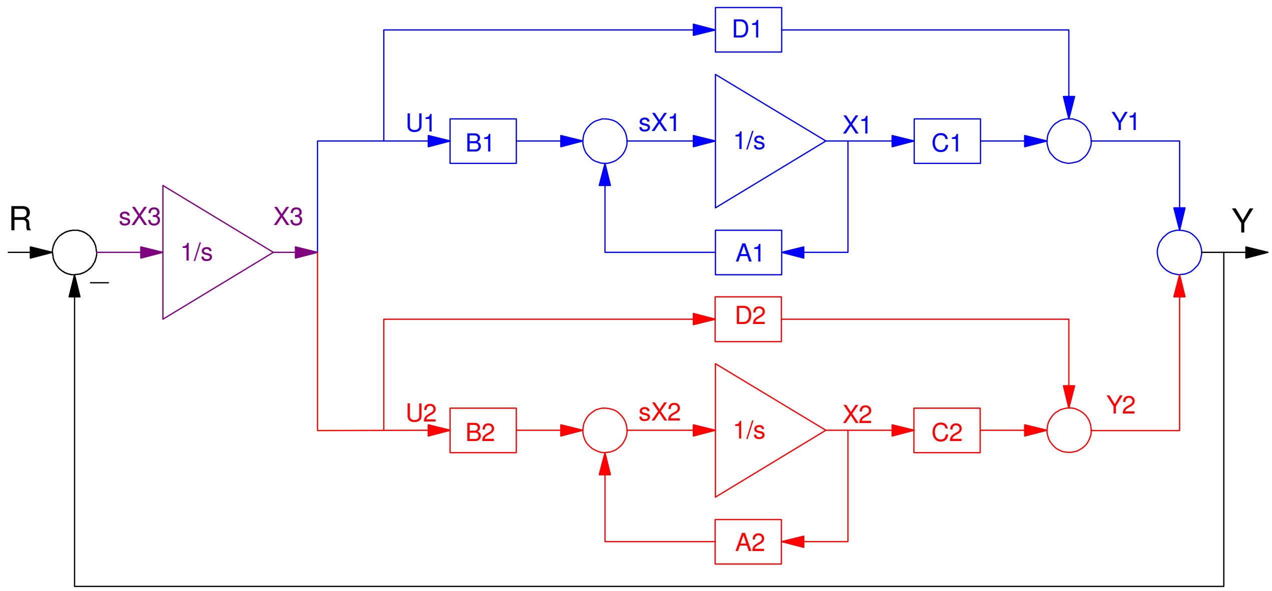
Net result

$$\begin{bmatrix} sZ_1 \\ sZ_2 \\ sZ_3 \\ sZ_4 \end{bmatrix} = \begin{bmatrix} -4 & 0 & -1 & -1 \\ 1 & -3 & 1 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \\ Z_4 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} V_0$$

$$Y = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} Z$$

Block Diagrams

5) Determine the state-space model the following system:



$$sX_1 = A_1X_1 + B_1X_3$$

$$sX_2 = A_2X_2 + B_2X_3$$

$$sX_3 = R - (C_2X_2 + D_2X_3 + C_1X_1 + D_1X_3)$$

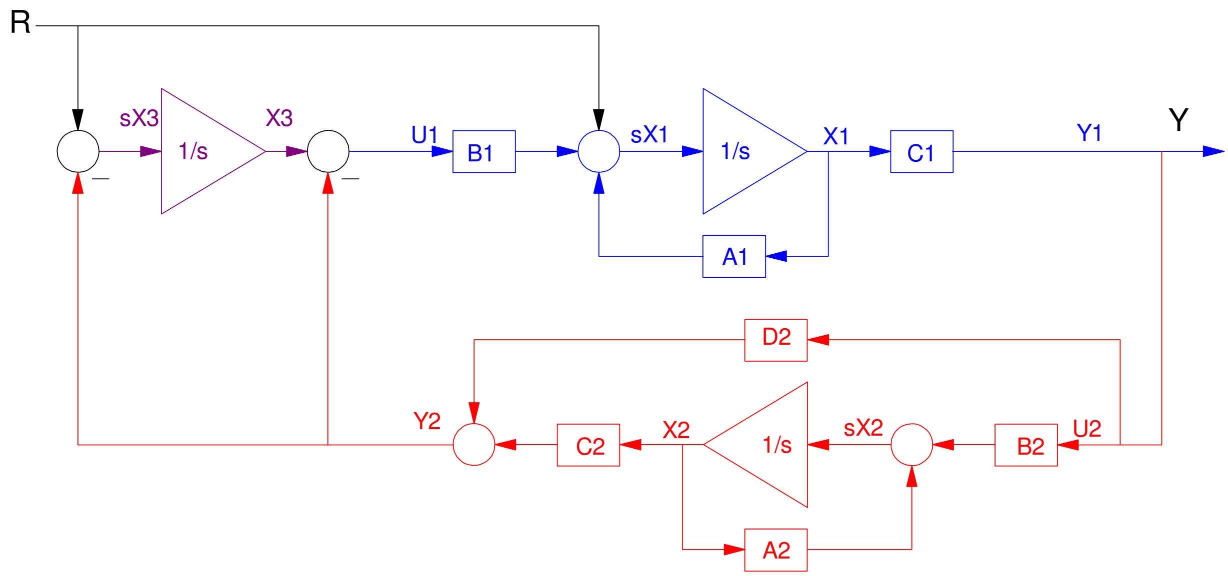
$$Y = C_1X_1 + D_1X_3 + C_2X_2 + D_2X_3$$

In matrix form

$$\begin{bmatrix} sX_1 \\ sX_2 \\ sX_3 \end{bmatrix} = \begin{bmatrix} A_1 & 0 & B_1 \\ 0 & A_2 & B_2 \\ -C_1 & -C_2 & -D_1 - D_2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} R$$

$$Y = \begin{bmatrix} C_1 & C_2 & D_1 + D_2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$$

6) Determine the state-space model for the following system:



$$sX_1 = A_1X_1 + B_1X_3 - B_1C_2X_2 - B_1D_2C_1X_1 + R$$

$$sX_2 = A_2X_2 + B_2C_1X_1$$

$$sX_3 = R - C_2X_2 - D_2C_1X_1$$

$$Y = C_1X_1$$

In matrix form

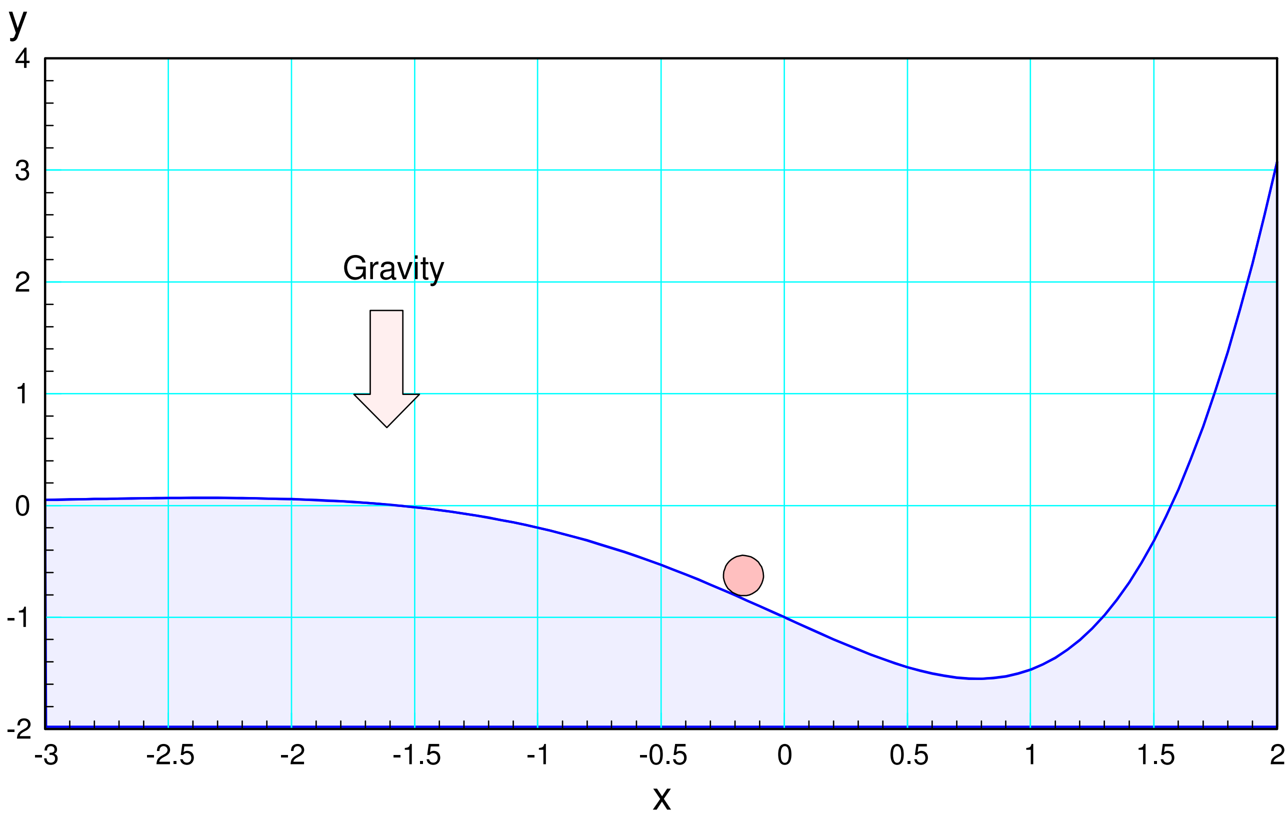
$$\begin{bmatrix} sX_1 \\ sX_2 \\ sX_3 \end{bmatrix} = \begin{bmatrix} A_1 - B_1D_2C_1 & -B_1C_2 & B_1 \\ B_2C_1 & A_2 & 0 \\ -D_2C_1 & -C_2 & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} R$$

$$Y = \begin{bmatrix} C_1 & 0 & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$$

LaGrangian Dynamics

A 1kg ball is rolling on a surface defined by:

$$y = -\exp(x) \cdot \cos(x)$$



7) Determine the kinetic and potential energy of this ball as a function of x: Gravity is in the -y direction.

Assuming a solid sphere:

The position is

$$y = -e^x \cdot \cos(x)$$

The potential energy is

$$PE = mgy$$

$$PE = -mg \cdot e^x \cdot \cos(x)$$

The velocity is

$$\dot{y} = -e^x \cdot \cos(x) \cdot \dot{x} + e^x \cdot \sin(x) \cdot \dot{x}$$

The kinetic energy is then (assuming a solid sphere)

$$KE = 0.7m(\dot{x}^2 + \dot{y}^2)$$

$$KE = 0.7m\dot{x}^2\left(1 + (-e^x \cdot \cos(x) + e^x \cdot \sin(x))^2\right)$$

$$KE = 0.7m\dot{x}^2\left(1 + e^{2x}\cos^2(x) - 2e^{2x}\cos(x)\sin(x) + e^{2x}\sin^2(x)\right)$$

$$KE = 0.7m\dot{x}^2(1 + e^{2x} - 2e^{2x}\cos(x)\sin(x))$$

8) Determine the dynamics for this ball as it rolls on this surface

Set up the LaGrangian

$$L = KE - PE$$

$$L = (0.7m\dot{x}^2(1 + e^{2x} - 2e^{2x}\cos(x)\sin(x))) - (-mg \cdot e^x \cdot \cos(x))$$

$$L = 0.7m\dot{x}^2(1 + e^{2x} - 2e^{2x}\cos(x)\sin(x)) + mg \cdot e^x \cdot \cos(x)$$

The dynamics are then

$$F = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \left(\frac{\partial L}{\partial x}\right)$$

$$F = 0 = \tfrac{d}{dt}(1.4m\dot{x}(1 + e^{2x} - 2e^{2x}\cos(x)\sin(x)))$$

$$-0.7m\dot{x}^2\left(2e^{2x} - 4e^{2x}\cos(x)\sin(x) + 2e^{2x}\sin^2(x) - 2e^{2x}\cos^2(x)\right)$$

$$+mg \cdot e^x \cdot \cos(x) - mg \cdot e^x \cdot \sin(x)$$

Taking the full derivative

$$0 = 1.4m\ddot{x}(1 + e^{2x} - 2e^{2x}\cos(x)\sin(x))$$

$$+1.4m\dot{x}\left(2\dot{x}e^{2x} - 4\dot{x}e^{2x}\cos(x)\sin(x) + 2e^{2x}\sin^2(x)\dot{x} - 2e^{2x}\cos^2(x)\dot{x}\right)$$

$$-0.7m\dot{x}^2\left(2e^{2x} - 4e^{2x}\cos(x)\sin(x) + 2e^{2x}\sin^2(x) - 2e^{2x}\cos^2(x)\right)$$

$$+mg \cdot e^x \cdot \cos(x) - mg \cdot e^x \cdot \sin(x)$$

This simplifies slightly

$$\sin^2 - \cos^2 = 2\sin^2 - 1$$

Using this identity

$$0 = 1.4m\ddot{x}(1 + e^{2x} - 2e^{2x}\cos(x)\sin(x))$$
$$+1.4m\dot{x}\left(-4\dot{x}e^{2x}\cos(x)\sin(x) + 4e^{2x}\sin^2(x)\dot{x}\right)$$
$$-0.7m\dot{x}^2\left(-4e^{2x}\cos(x)\sin(x) + 4e^{2x}\sin^2(x)\right)$$
$$+mg \cdot e^x \cdot \cos(x) - mg \cdot e^x \cdot \sin(x)$$

Simplifying further

$$0 = 1.4m\ddot{x}(1 + e^{2x} - 2e^{2x}\cos(x)\sin(x))$$
$$+2.8m \cdot e^{2x} \cdot \sin(x) \cdot (\cos(x) - \sin(x)) \cdot \dot{x}^2$$
$$+mg \cdot e^x \cdot (\cos(x) - \sin(x))$$

This allows you to calculate the acceleration at any given time

$$\ddot{x} = \left(\frac{2.8 \cdot e^{2x} \cdot \sin(x) \cdot (\cos(x) - \sin(x)) \cdot \dot{x}^2 + g \cdot e^x \cdot (\cos(x) - \sin(x))}{1.4\left(1 + e^{2x} - 2e^{2x}\cos(x)\sin(x)\right)} \right)$$

Note that the mass cancels - so the dynamics do not depend upon the mass.