Homework #10: ECE 461/661

z-Transforms, s to z conversion, Root Locus in the z-Domain. Due Monday, November 13th

z-Transforms

1) Determine the difference equation that relates X and Y

$$Y = \left(\frac{0.05z(z-1)}{(z-0.9)(z-0.8)(z-0.5)}\right)X$$

Cross multiply and multiply out the polynomials

$$(z^3 - 2.2z^2 + 1.57z - 0.36)Y = 0.05(z^2 - z)X$$

Note that 'zY' means 'the next value of y(k)' or 'y(k+1)'

$$y(k+3) - 2.2y(k+2) + 1.57y(k+1) - 0.36y(k) = 0.05(x(k+2) - x(k+1))$$

2) Determine y(k) assuming

$$Y = \left(\frac{0.05z(z-1)}{(z-0.9)(z-0.8)(z-0.5)}\right)X \qquad x(t) = 2\cos(4t) + 3\sin(4t)$$

$$T = 0.01$$

This is a phasor problem

$$X = 2 - j3$$

$$s = j4$$

$$z = e^{sT} = e^{j0.04}$$

$$Y = \left(\frac{0.05z(z-1)}{(z-0.9)(z-0.8)(z-0.5)}\right)_{z=e^{j0.04}} \cdot (2 - j3) = 0.6625 - j0.0088$$

meaning

$$y(t) = 0.6625\cos(4t) + 0.0088\sin(4t)$$

3) Determine y(k) assuming

$$Y = \left(\frac{0.05z(z-1)}{(z-0.9)(z-0.8)(z-0.5)}\right)X \qquad x(k) = u(k)$$

Use z-transforms

$$Y = \left(\frac{0.05z(z-1)}{(z-0.9)(z-0.8)(z-0.5)}\right)\left(\frac{z}{z-1}\right)$$

Pull out a z

$$Y = \left(\frac{0.05z(z-1)}{(z-1)(z-0.9)(z-0.8)(z-0.5)}\right)z$$

Do a partial fraction expansion

$$Y = \left(\left(\frac{0}{z-1}\right) + \left(\frac{1.125}{z-0.9}\right) + \left(\frac{-1.333}{z-0.8}\right) + \left(\frac{0.2083}{z-0.5}\right)\right)z$$

Multiply through by z

$$Y = \left(\frac{0z}{z-1}\right) + \left(\frac{1.125z}{z-0.9}\right) + \left(\frac{-1.333z}{z-0.8}\right) + \left(\frac{0.2083z}{z-0.5}\right)$$

Take the inverse z-transform

$$y(k) = \left(0 + 1.125(0.9)^k - 1.333(0.8)^k + 0.2083(0.5)^k\right)u(k)$$

s to z conversion

4) Determine the discrete-time equivalent of G(s). Assume T = 0.1 seconds

$$G(s) = \left(\frac{2331}{(s+2.6338)(s+30.2062)(s+53.7896)} \right)$$

```
>> T = 0.1;
>> s1 = -2.6338;
>> s2 = -30.2062;
>> s3 = -53.7896;

>> z1 = exp(s1*T)
   z1 =    0.7684
>> z2 = exp(s2*T)
   z2 =    0.0488
>> z3 = exp(s3*T)
   z3 =    0.0046

>> Gz = zpk([],[z1,z2,z3],1,0.1)

                  1
-----------------------------------
(z-0.7684) (z-0.04877) (z-0.004613)

Sampling time (seconds): 0.1

>> Gs = zpk([],[-2.6338,-30.2062,-57.7896],2331)

            2331
-----------------------------
(s+2.634) (s+30.21) (s+57.79)

>> DC = evalfr(Gs,0)

DC =    0.5070

>> k = evalfr(Gs,0) / evalfr(Gz,1)

k =    0.1112

>> Gz = zpk([],[z1,z2,z3],k,0.1)

              0.11116
-----------------------------------
(z-0.7684) (z-0.04877) (z-0.004613)

Sampling time (seconds): 0.1
```

(optional) You can make the model a little more accurate by adding zeros at z=0.

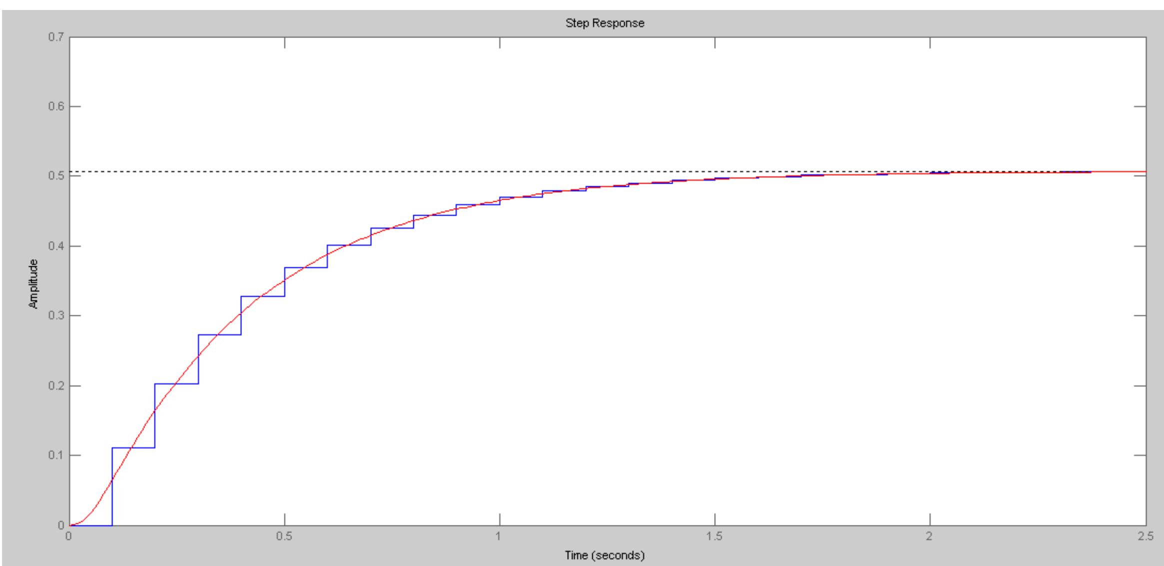
- You can do this by trial-and-error (plot the step responses of G(s) and G(z). Add zeros at z=0 until the delay is 'just right')
- You can calculate the number of zeros needed as well.

Calculations: Pick a frequency like s = j. Match the phase at that frequency.

```
>> s = j;
>> T = 0.1;
>> z = exp(s*T);

>> Ps = angle(evalfr(Gs,s))*180/pi

Ps =  -23.6782

>> Pz = angle(evalfr(Gz,z))*180/pi

Pz =  -35.5600

>> angle(z)*180/pi

ans =    5.7296

>> n = (Ps - Pz) / 5.7296

n =    2.0738
```

You need 2.07 zeros at z=0 to make the phase match up at s = j. Let n=2

```
>> Gz = zpk([0,0],[z1,z2,z3],k,0.1)

            0.1112 z^2
-----------------------------------
(z-0.7684) (z-0.04877) (z-0.004613)

Sampling time (seconds): 0.1
>> step(Gz)
>> hold on
>> t = [0:0.01:3.5]';
>> y = step(Gs,t);
>> plot(t,y,'r')
```



5) Determine the discrete-time equivalent of G(s). Assume T = 0.01 seconds

First, convert the poles from the s-plane to the z-plane:

```
>> s = [-2.6338, -30.2062, -53.7896];
>> T = 0.01;
>> z = exp(s*T)

z =    0.9740    0.7393    0.5840

>> Gs = zpk([],[-2.6338,-30.2062,-57.7896],2331)

            2331
----------------------------
(s+2.634) (s+30.21) (s+57.79)

>> Gz = zpk([],z,1,0.01)

              1
------------------------------
(z-0.974) (z-0.7393) (z-0.584)

Sampling time (seconds): 0.01
```

Find k to match the DC gain:

```
>> k = evalfr(Gs,0) / evalfr(Gz,1)

k =    0.0014
>> Gz = zpk([],z,k,0.01)

          0.0014294
------------------------------
(z-0.974) (z-0.7393) (z-0.584)

Sampling time (seconds): 0.01
```

Add zeros at z=0 to match the phase shift at s = j

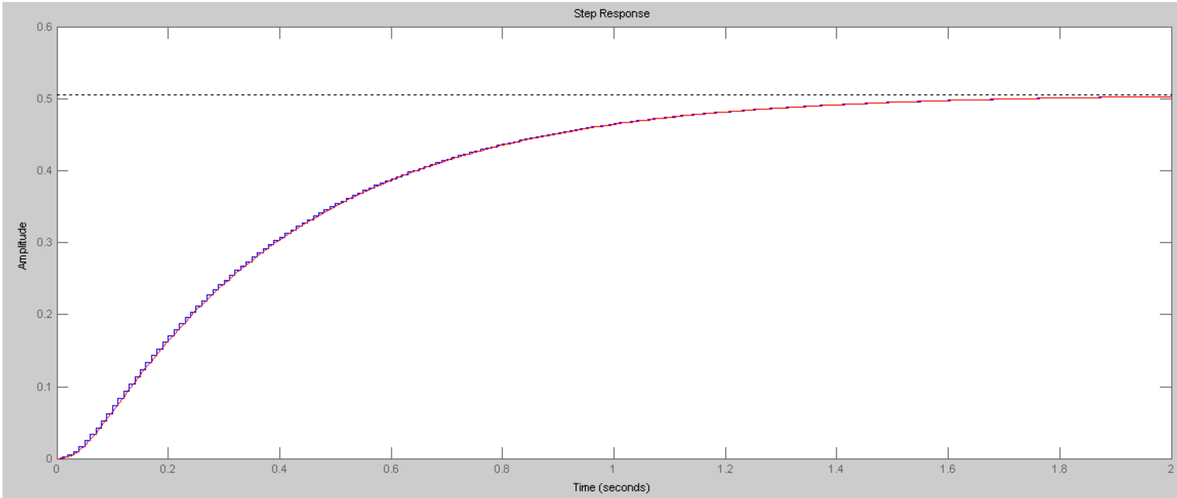
```
>> s = j;
>> z = exp(s*T);
>> n = (angle(evalfr(Gs,s)) - angle(evalfr(Gz,z))) / angle(z)

n =    1.7006
```

Round to two zeros. This gives G(z):

```
>> Gz = zpk([0,0],z,k,0.01)

        0.0014294 z^2
------------------------------
(z-0.974) (z-0.7393) (z-0.584)

Sampling time (seconds): 0.01
```

Checking the step response of G(s) and G(z):

```
>> step(Gz)
>> hold on
>> plot(t,y,'r')
>> xlim([0,2])
>> ylim([0,0.6])
>>
```


Step Response
Amplitude
Time (seconds)
0.6
0.5
0.4
0.3
0.2
0.1
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2

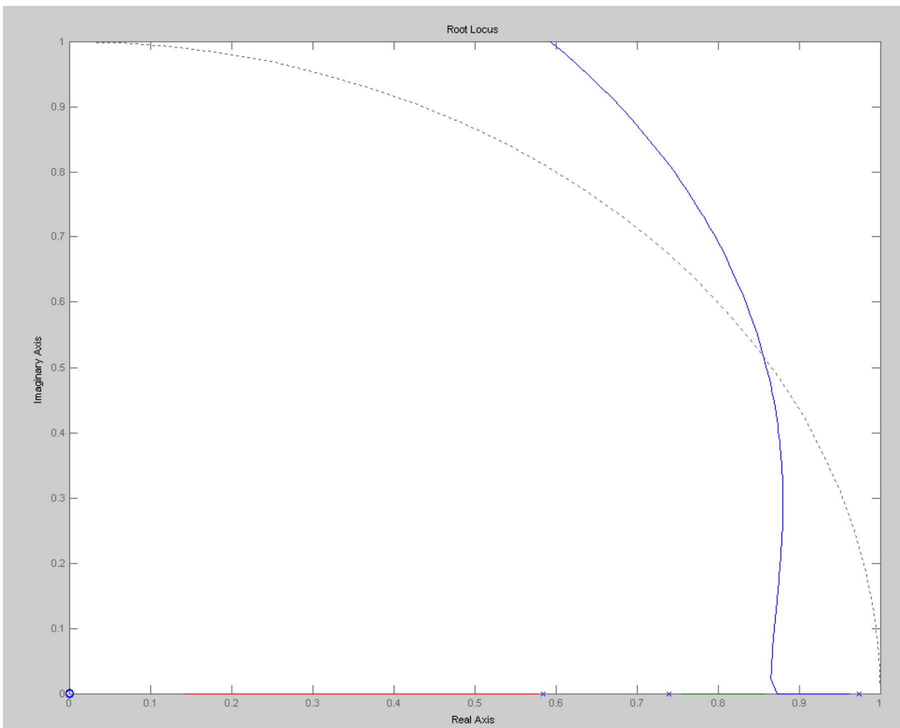
Root Locus in the z-Domain

Assume T = 0.01 seconds.

$$G(s) = \left(\frac{2331}{(s+2.6338)(s+30.2062)(s+53.7896)} \right)$$

6) Draw the root locus for G(z)

```
            0.0014294 z^2
------------------------------
(z-0.974) (z-0.7393) (z-0.584)

Sampling time (seconds): 0.01
>> k = logspace(-2,2,200)';
>> rlocus(Gz,k);
>> ylim([0,1])
>> xlim([0,1])
```

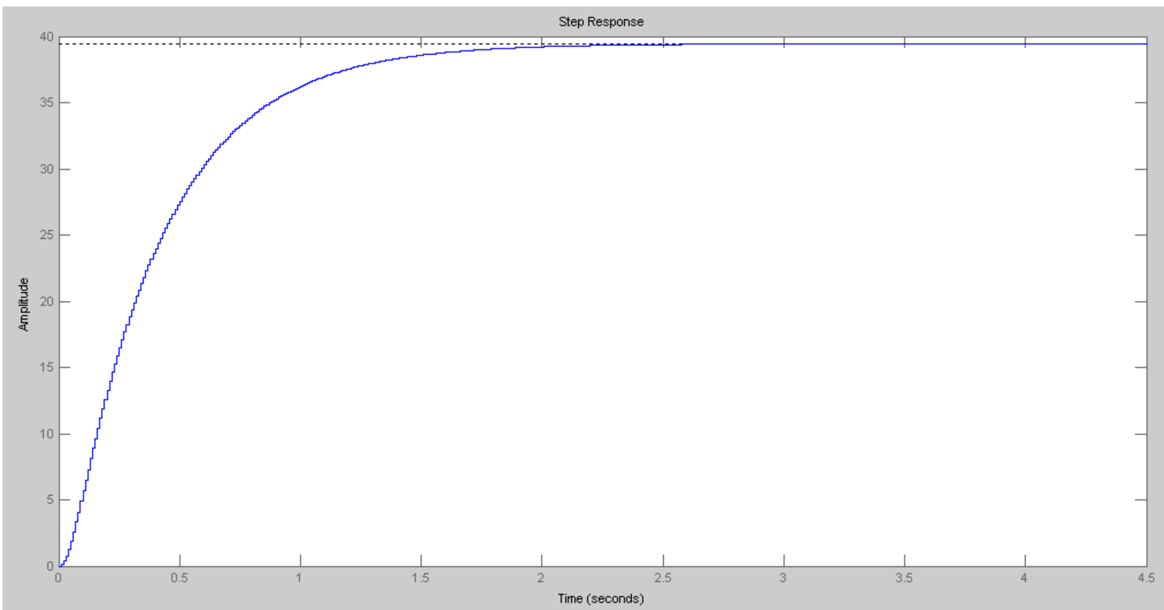


7) Find k for no overshoot in the step response

- Simulate the closed-loop system's step response

At the breakaway point, z = 0.86

```
>> z = 0.86;
>> evalfr(Gz,z)

ans =  -21.6517

>> k = 1/abs(ans)

k =    0.0462

>> Gcl = minreal(Gz*k / (1+Gz*k))


         0.0051359 z^2
------------------------------
(z-0.8703) (z-0.86) (z-0.5618)

Sampling time (seconds): 0.01

>> step(Gz)
```

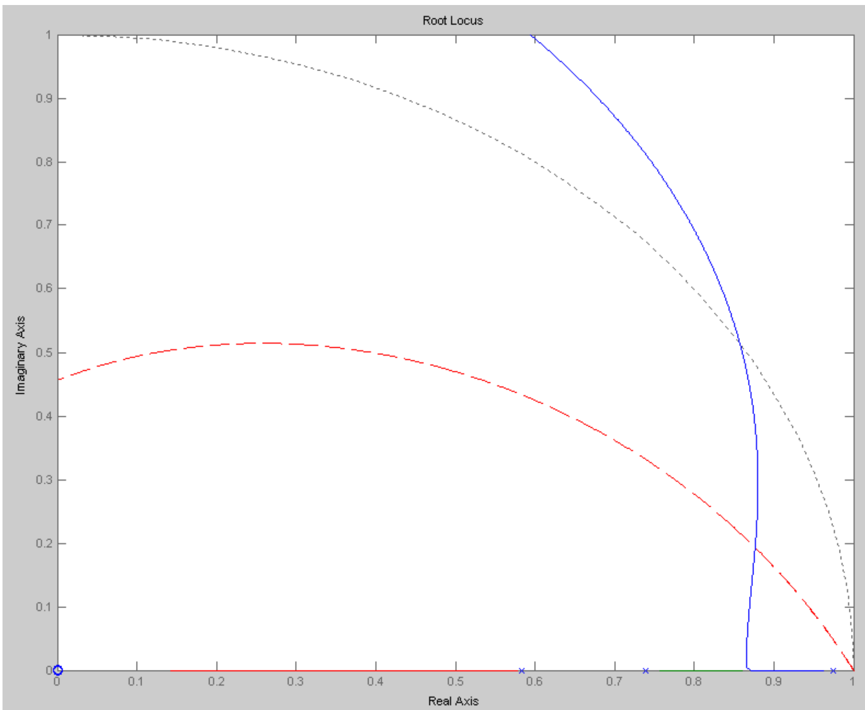


8) Find k for 20% overshoot for a step response (damping ratio = 0.4559)

- Simulate the closed-loop system's step response

First, plot the root locus along with the damping line:

```
>> k = logspace(-2,2,2000)';
>> rlocus(Gz,k);

>> s = (-1+j*2) * [0:0.1:100]';
>> z = exp(s*T);
>> plot(real(z),imag(z),'r--')
>> xlim([0,1])
>> ylim([0,1])
```



Find the spot on the root locus that intersects the damping line:

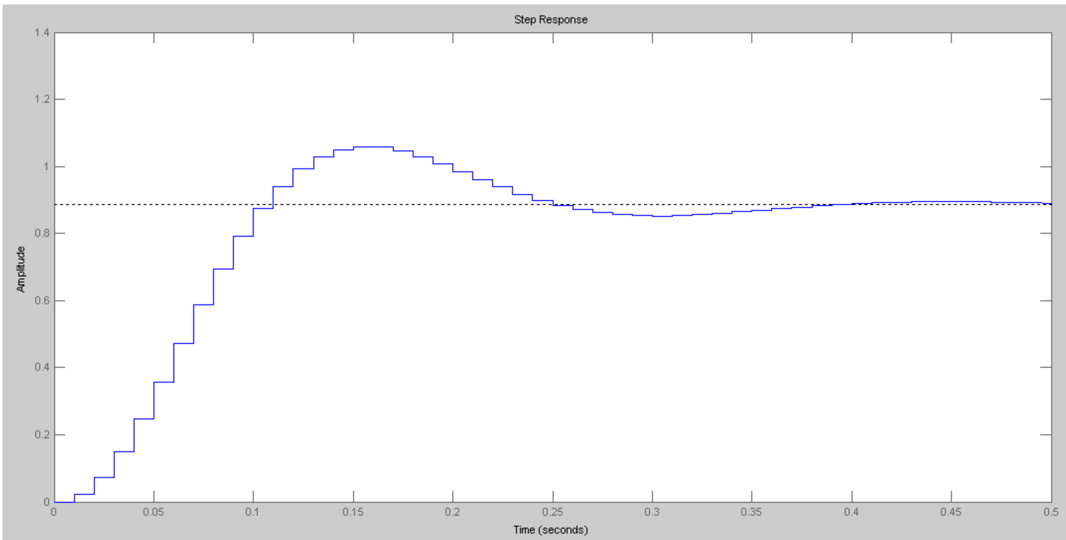
```
z = 0.8674 + j0.1928
```

Pick k to make the gain one at this point

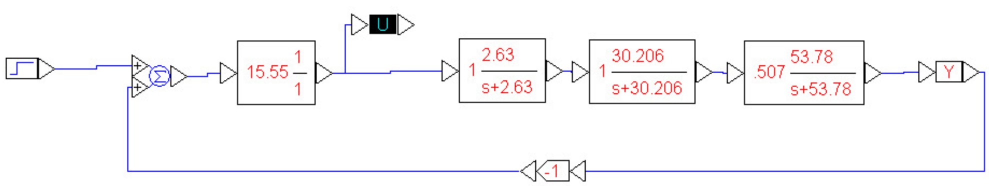
```
>> evalfr(Gz,z)

ans =  -0.0643 + 0.0000i

>> k = 1 / abs(ans)

k =   15.5576
```

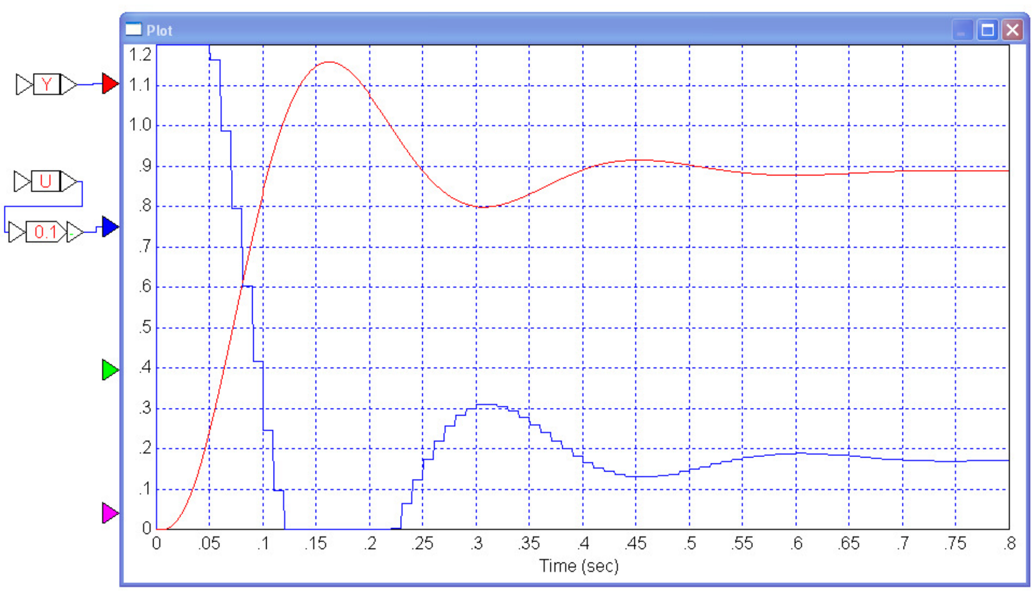
Plot the closed-loop step response:

```
>> Gcl = minreal(Gz*k / (1+Gz*k))

            0.022238 z^2
  ----------------------------------
  (z-0.5222) (z^2 - 1.753z + 0.8053)

Sampling time (seconds): 0.01
>> step(Gcl)
>>
```



or in VisSim

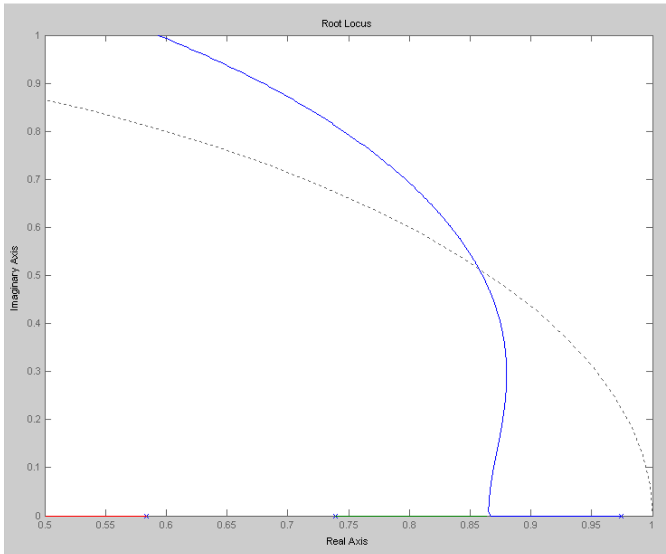




9) Find k for a damping ratio of 0.00

- Simulate the closed-loop system's step response

Find the spot on the root locus that crosses the unit circle



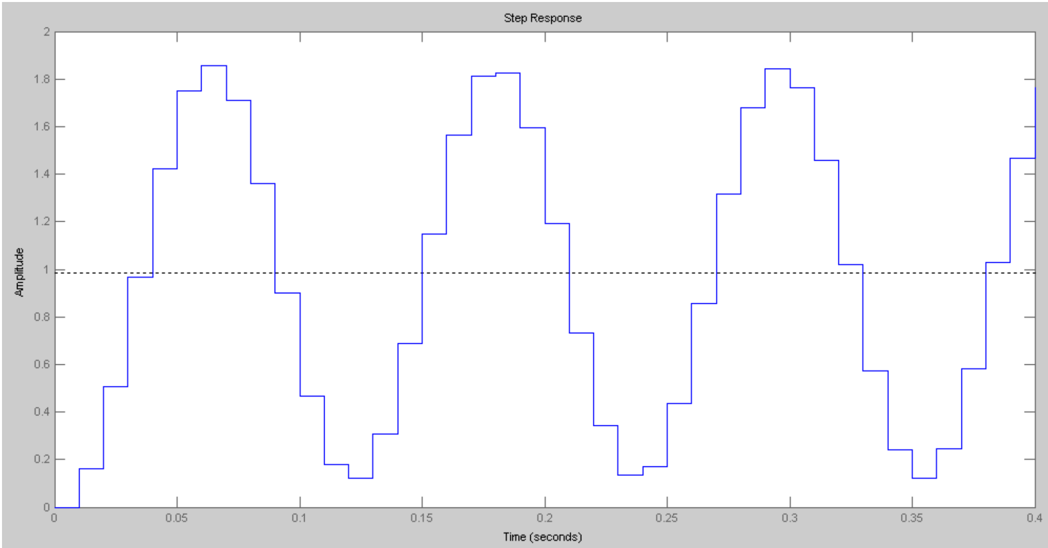
$z = 0.8571 + j0.5142$

Pick k to make the gain one at this point:

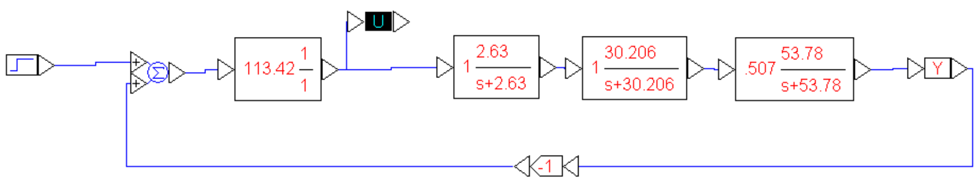
```
>> z = 0.8571 + j*0.5142;
>> evalfr(Gz,z)

ans =  -0.0088 + 0.0000i

>> k = 1 / abs(ans)

k =  113.4194
```

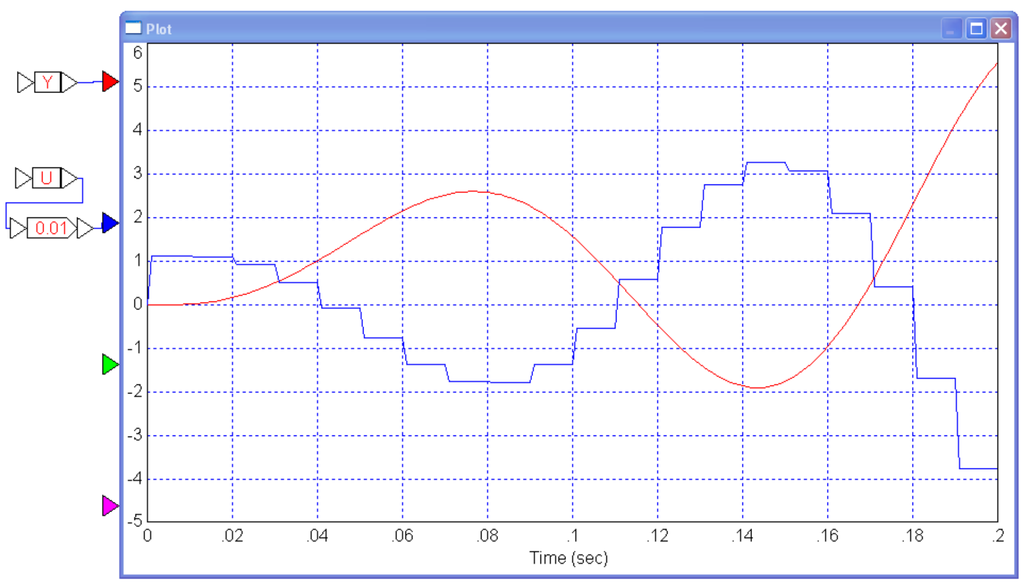
Plot the closed-loop step response:

```
>> Gcl = minreal(Gz*k / (1+Gz*k))

Zero/pole/gain:
           0.16212 z^2
----------------------------------
(z-0.4209) (z^2 - 1.714z + 0.9991)

Sampling time (seconds): 0.01
>> step(Gcl)
>> xlim([0,0.4])
```


Step Response
Amplitude
Time (seconds)
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4

or in VisSim


U
113.42
1
1
1
2.63
s+2.63
1
30.206
s+30.206
.507
53.78
s+53.78
Y
-1


Plot
Y
U
0.01
Time (sec)
6
5
4
3
2
1
0
-1
-2
-3
-4
-5
0
.02
.04
.06
.08
.1
.12
.14
.16
.18
.2

Note: There's a more accurate way of doing this.

- Model G(s) as G(s) (no change)
- Model the sampling as a 1/2 sample delay
- Model K(z) as k (no change)

Search along the s = jw axis until the angles add up to 180 degrees

$$\left(\left(\frac{2331}{(s+2.6338)(s+30.2062)(s+53.7896)}\right) \cdot \exp\left(\frac{-sT}{2}\right) \cdot k\right)_{s=j\omega} = x\angle 180^0$$

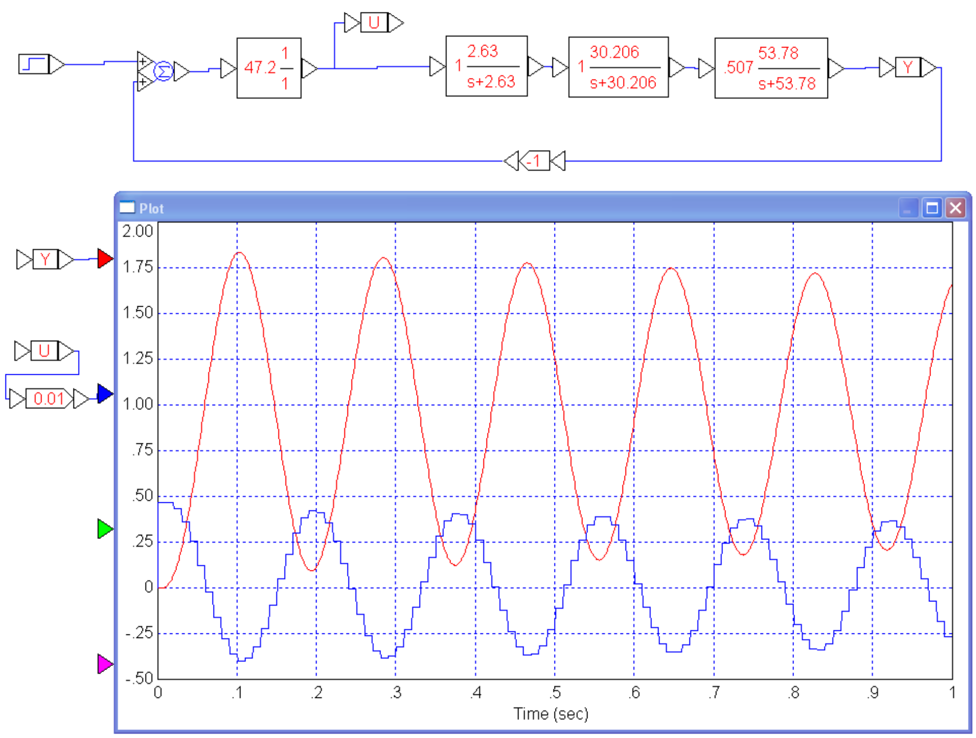
The results is

- s = 0 + j36.0398
- z = 0.9358 + j0.3526

$$\left(\left(\frac{2331}{(s+2.6338)(s+30.2062)(s+53.7896)}\right) \cdot \exp\left(\frac{-sT}{2}\right) \cdot k\right)_{s=j36.0398} = 0.0212\angle 180^0$$

so

$$k = \frac{1}{0.0212} = 47.20$$



Which method is best?

- The one that gives the correct answqer