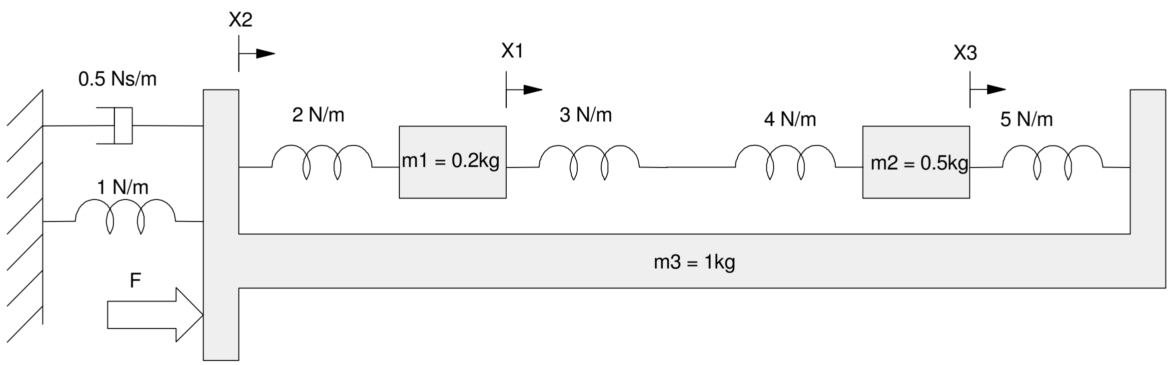
Homework #6: ECE 461/661

Mass-Spring Systems, Rotational Systems.   Due Monday, September 30th

Mass Spring systems

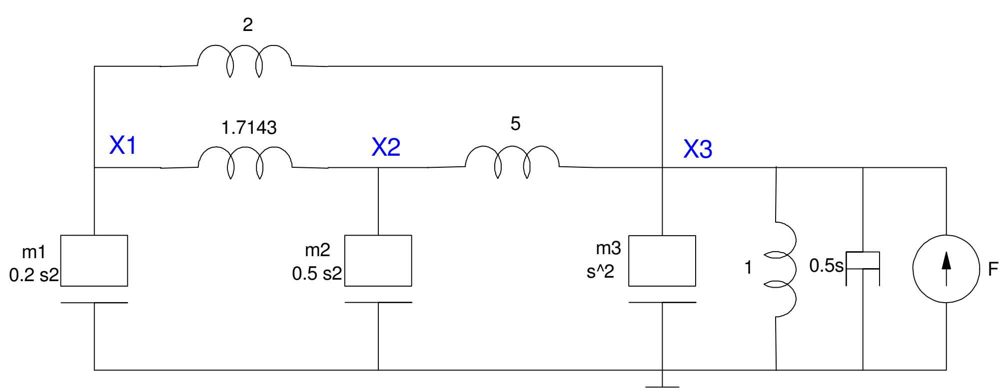
1) (20pt) Draw the circuit equivalent for the following mass-spring systems.
   - Express the dynamics in state-space form
   - Find the transfer function from F to X2
   - Plot the step response from F to X2



Add the 3N/m and 4N/m admittances in series

$$K = \left(\frac{1}{3} + \frac{1}{4}\right)^{-1} = 1.714 N/m$$

Draw the circuit equivalent



Write the node equations

$$(0.2s^2 + 1.7143 + 2)x_1 - (1.7143)x_2 - (2)x_3 = 0$$

$$(0.5s^2 + 1.7143 + 5)x_2 - (1.7143)x_1 - (5)x_3 = 0$$

$$(s^2 + 2 + 5 + 1 + 0.5s)x_3 - (2)x_1 - (5)x_2 = F$$

Solve for the highest derivatives

$$s^2 x_1 = -18.57 x_1 + 8.57 x_2 + 10 x_3$$

$$s^2 x_2 = 3.43 x_1 - 13.43 x_2 + 10 x_3$$

$$s^2 x_3 = 2 x_1 + 5 x_2 - 8 x_3 - 0.5 s x_3 + F$$

Place in matrix form

$$s\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ sx_1 \\ sx_2 \\ sx_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -18.57 & 8.57 & 10 & 0 & 0 & 0 \\ 3.43 & -13.43 & 10 & 0 & 0 & 0 \\ 2 & 5 & -8 & 0 & 0 & -0.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ sx_1 \\ sx_2 \\ sx_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} F$$

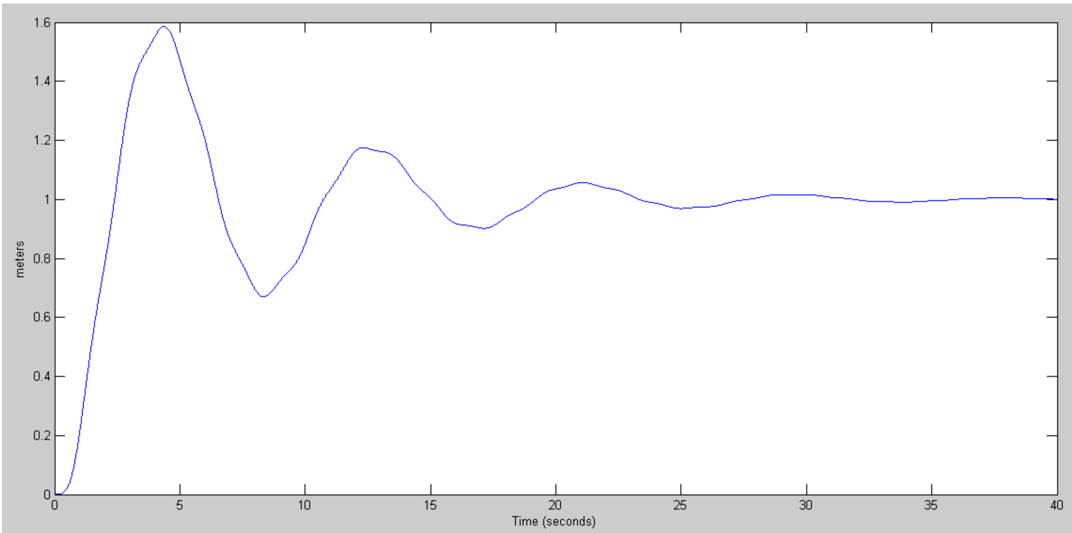
$$y = x_2 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} X + [0] F$$

Throw into Matlab and solve

```
>> Z = zeros(3,3);
>> I = eye(3,3);
>> K = [-18.57,8.57,10;3.43,-13.43,10;2,5,-8];
>> B = [0,0,0;0,0,0;0,0,-0.5];
>> A = [Z,I ; K,B]

         0         0         0    1.0000         0         0
         0         0         0         0    1.0000         0
         0         0         0         0         0    1.0000
  -18.5700    8.5700   10.0000         0         0         0
    3.4300  -13.4300   10.0000         0         0         0
    2.0000    5.0000   -8.0000         0         0   -0.5000

>> B = [0;0;0;0;0;1]
>> C = [0,1,0,0,0,0];
>> D = [0];
>> G = ss(A,B,C,D);
>> zpk(G)

                     10 (s^2 + 22)
---------------------------------------------------------
(s^2 + 0.2807s + 0.576) (s^2 + 0.2193s + 17.36) (s^2 + 22)
```

Plotting the step response

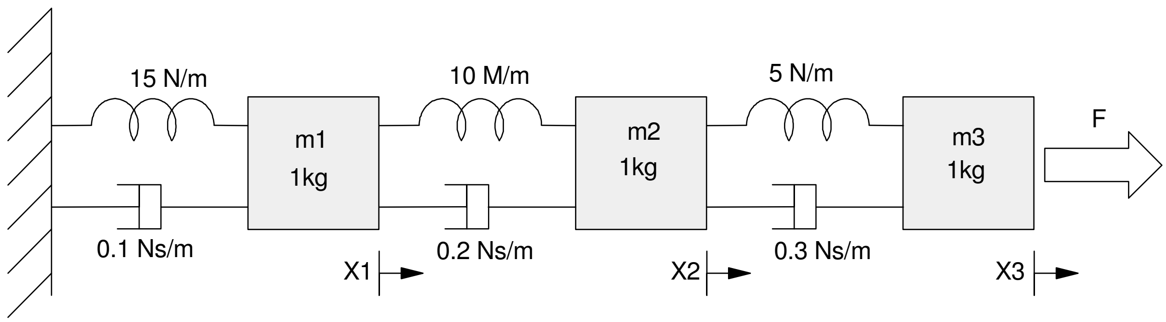
```
>> t = [0:0.01:10]';
>> y = step(G,t);
>> plot(t,y)
>> t = [0:0.01:40]';
>> y = step(G,t);
>> plot(t,y)
>> xlabel('Time (seconds)')
>> ylabel('meters')
```



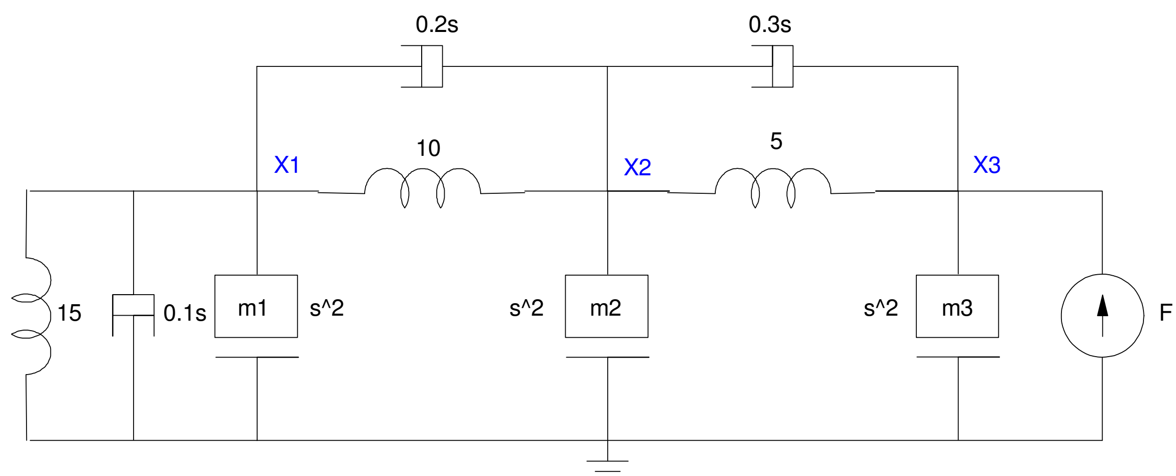
2) (20pt) Draw the circuit equivalent for the following mass-spring systems.

- Express the dynamics in state-space form
- Find the transfer function from F to X3

Plot the step response from F to X3



Draw the circuit equivalent



Write the voltage node equations

$$(s^2 + 0.1s + 15 + 0.2s + 10)x_1 - (0.2s + 10)x_2 = 0$$

$$(s^2 + 0.2s + 10 + 0.3s + 5)x_2 - (0.2s + 10)x_1 - (0.3s + 5)x_3 = 0$$

$$(s^2 + 0.3s + 5)x_3 - (0.3s + 5)x_2 = F$$

Solve for the highest derivatives

$$s^2 x_1 = -(0.3s + 25)x_1 + (0.2s + 10)x_2$$

$$s^2 x_2 = (0.2s + 10)x_1 - (0.5s + 15)x_2 + (0.3s + 5)x_3$$

$$s^2 x_3 = (0.3s + 5)x_2 - (0.3s + 5)x_3 + F$$

Place in matrix form

$$s\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ sx_1 \\ sx_2 \\ sx_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -25 & 10 & 0 & -0.3 & 0.2 & 0 \\ 10 & -15 & 5 & 0.2 & -0.5 & 0.3 \\ 0 & 5 & -5 & 0 & 0.3 & -0.3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ sx_1 \\ sx_2 \\ sx_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} F$$

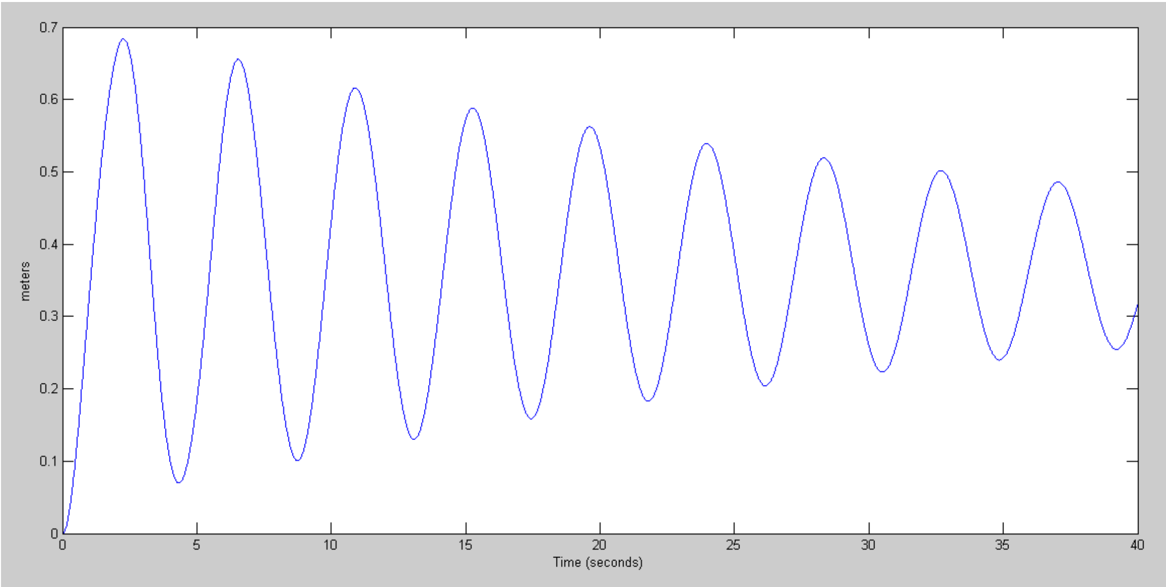
$$y = x_3 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} X + [0]F$$

Solve in Matlab

```
>> K = [-25,10,0;10,-15,5;0,5,-5];
>> B = [-0.3,0.2,0;0.2,-0.5,0.3;0,0.3,-0.3];
>> A = [Z,I ; K,B]

         0          0          0     1.0000          0          0
         0          0          0          0     1.0000          0
         0          0          0          0          0     1.0000
  -25.0000    10.0000          0    -0.3000     0.2000          0
   10.0000   -15.0000     5.0000     0.2000    -0.5000     0.3000
         0     5.0000    -5.0000          0     0.3000    -0.3000

>> eig(A)

  -0.2851 + 5.5926i
  -0.2851 - 5.5926i
  -0.2365 + 3.3815i
  -0.2365 - 3.3815i
  -0.0284 + 1.4424i
  -0.0284 - 1.4424i

>> B = [0;0;0;0;0;1];
>> C = [0,0,1,0,0,0];
>> D = 0;
>> G = ss(A,B,C,D);
>> zpk(G)


            (s^2 + 0.2661s + 8.832) (s^2 + 0.5339s + 31.14)
  -----------------------------------------------------------------------

  (s^2 + 0.05688s + 2.081) (s^2 + 0.4729s + 11.49) (s^2 + 0.5702s + 31.36)
```

Plot the step response

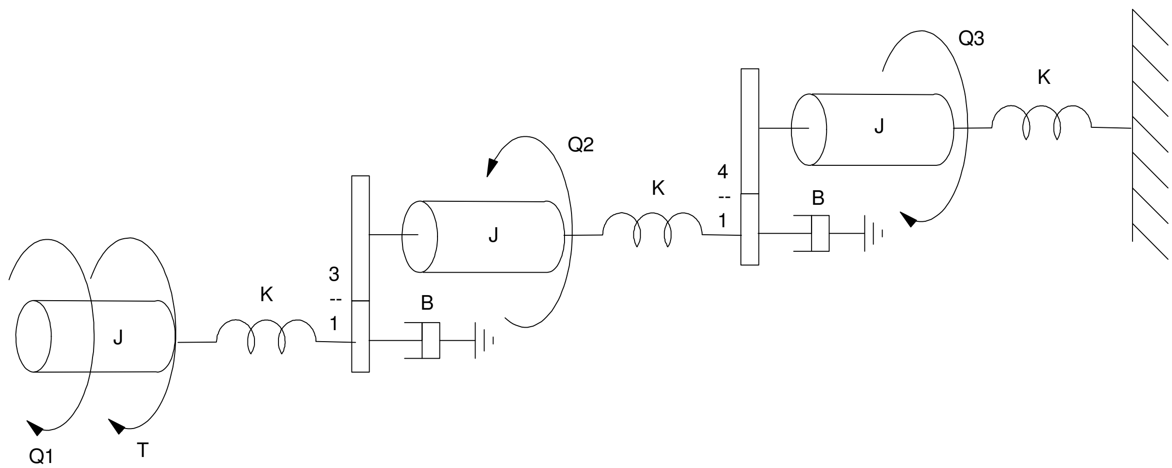
```
>> t = [0:0.01:40]';
>> y = step(G,t);
>> plot(t,y)
>> xlabel('Time (seconds)')
>> ylabel('meters')
>>
```



Rotational Systems

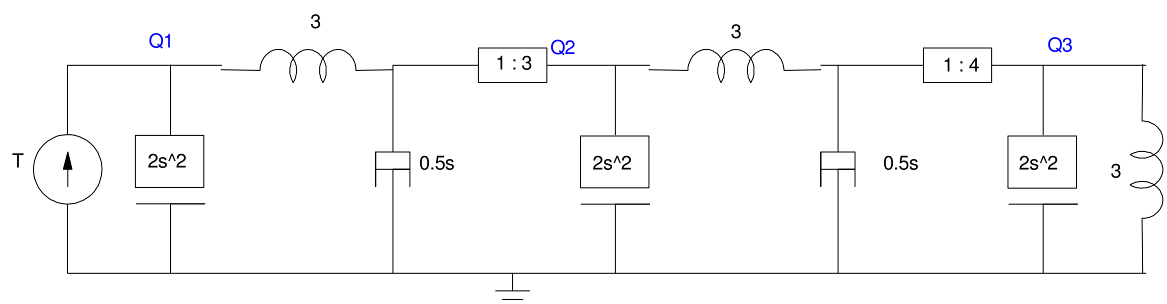
3) Draw the circuit equivalent for the following rotational system.

- Express the dynamics in state-space form
- Find the transfer function from T to Q1
- Plot the step response from T to Q1

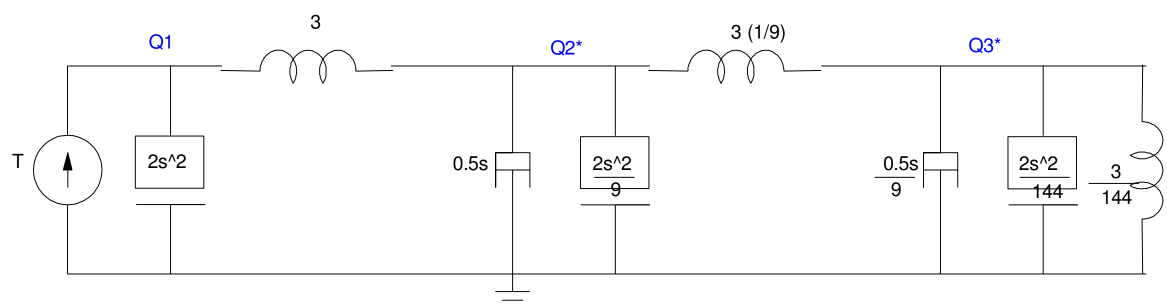


Problem 3: J = 2.0 Kg m / s^2, B = 0.5 Ns/m, K = 3 Nm/rad

Draw the circuit equivalent



Remove the gears



Write the voltage node equations

$$(2s^2+3)\theta_1-(3)\theta_2^*=T$$

$$\left(\tfrac{2}{9}s^2+0.5s+3+\tfrac{3}{9}\right)\theta_2^*-(3)\theta_1-\left(\tfrac{3}{9}\right)\theta_3^*=0$$

$$\left(\tfrac{2}{144}s^2+\tfrac{0.5}{9}s+\tfrac{3}{144}+\tfrac{3}{9}\right)\theta_3^*-\left(\tfrac{3}{9}\right)\theta_2^*=0$$

Solve for the highest derivatives (drop the asterisks: q2 and q3 are scaled of the actual q2 and q3)

$$s^2\theta_1=-(1.5)\theta_1+(1.5)\theta_2+(0.5)T$$

$$s^2\theta_2=(13.5)\theta_1-(2.25s\theta_2+15\theta_2)+(1.5)\theta_3$$

$$s_3\theta_3=(24)\theta_2-(4s\theta_3+25.5)\theta_3$$

Place in matrix form

$$s\begin{bmatrix}\theta_1\\ \theta_2\\ \theta_3\\ s\theta_1\\ s\theta_2\\ s\theta_3\end{bmatrix}=\begin{bmatrix}0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1\\ -1.5&1.5&0&0&0&0\\ 13.5&-15&1.5&0&-2.25&0\\ 0&24&-25.5&0&0&-4\end{bmatrix}s\begin{bmatrix}\theta_1\\ \theta_2\\ \theta_3\\ s\theta_1\\ s\theta_2\\ s\theta_3\end{bmatrix}+\begin{bmatrix}0\\ 0\\ 0\\ 0.5\\ 0\\ 0\end{bmatrix}T$$

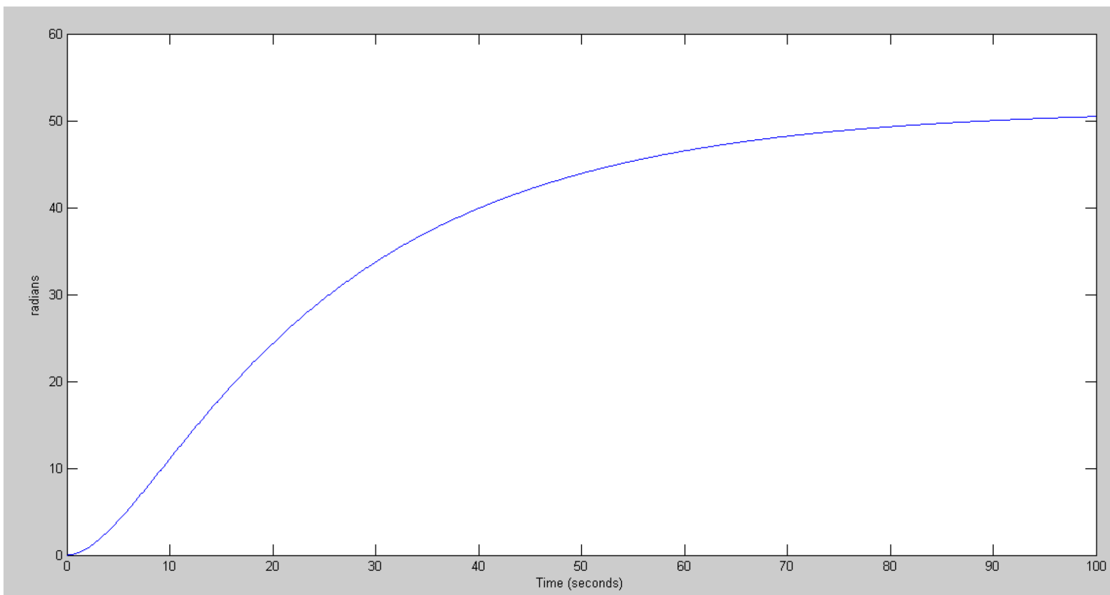
Solve in Matlab with the output being q1

```
>> Z = zeros(3,3);
>> I = eye(3,3);
>> K = [-1.5,1.5,0;13.5,-15,1.5;0,24,-25.5];
>> B = [0,0,0;0,-2.25,0;0,0,-4];
>> A = [Z,I ; K,B]

          0          0          0     1.0000          0          0
          0          0          0          0     1.0000          0
          0          0          0          0          0     1.0000
    -1.5000     1.5000          0          0          0          0
    13.5000   -15.0000     1.5000          0    -2.2500          0
          0    24.0000   -25.5000          0          0    -4.0000

>> eig(A)

  -1.8060 + 4.8955i
  -1.8060 - 4.8955i
  -1.1923 + 3.4941i
  -1.1923 - 3.4941i
  -0.2101
  -0.0433

>> B = [0;0;0;0.5;0;0];
>> C = [1,0,0,0,0,0];
>> D = 0;
>> G = ss(A,B,C,D);
```

```
>> zpk(G)


          0.5 (s^2 + 2.603s + 12.68) (s^2 + 3.647s + 27.33)
--------------------------------------------------------------------

(s+0.2101) (s+0.04329) (s^2 + 2.385s + 13.63) (s^2 + 3.612s + 27.23)


>>
```

Plotting the step response in Matlab

```
>> t = [0:0.1:100]';
>> y = step(G,t);
>> plot(t,y)
>> xlabel('Time (seconds)')
>> ylabel('radians')>>
```




```
>>
```