Homework #4: ECE 461 / 661

1st and 2nd Order Approximations. Due Monday, September 16th

LaPlace Transforms

1) Assume X and Y are related by the following transfer function

$$Y = \left(\frac{7s+2}{(s+2)(s+6)(s+8)}\right)X$$

a) What is the differential equation relating X and Y?

Cross multiply

$$(s+2)(s+6)(s+8)Y = (7s+2)X$$

$$(s^3 + 16s^2 + 76s + 96)Y = (7s+2)X$$

'sY' means 'the derivative of y(t)'

$$\boxed{y''' + 16y'' + 76y' + 96y = 7x' + 2x}$$

b) Determine y(t) assuming

$$x(t) = 2\cos(5t) + 3\sin(5t)$$

Use phasors

$$X = 2 - j3$$

$$s = j5$$

$$Y = \left(\frac{7s+2}{(s+2)(s+6)(s+8)}\right)_{s=j5} \cdot (2 - j3)$$

$$Y = -0.1068 - j0.3001$$

meaning

$$\boxed{y(t) = -0.1068\cos(5t) + 0.3001\sin(5t)}$$

c) Determine y(t) assuming x(t) is a unit step input

$$Y = \left(\frac{7s+2}{(s+2)(s+6)(s+8)}\right)\left(\frac{1}{s}\right)$$

Use partial fractions

$$Y = \left(\frac{0.0208}{s}\right) + \left(\frac{0.25}{s+2}\right) + \left(\frac{-0.8333}{s+6}\right) + \left(\frac{0.5625}{s+8}\right)$$

meaning

$$y(t) = (0.0208 + 0.25e^{-2t} - 0.8333e^{-6t} + 0.5625e^{-8t})u(t)$$

Matlab Code:

```
>> G = zpk([-2/7],[0,-2,-6,-8],7)

   7 (s+0.2857)
-------------------
s (s+2) (s+6) (s+8)

>> s = 0 + 1e-9;
>> evalfr(G, s) * s

    0.0208

>> s = -2 + 1e-9;
>> evalfr(G, s) * (s+2)

    0.2500

>> s = -6 + 1e-9;
>> evalfr(G, s) * (s+6)

   -0.8333

>> s = -8 + 1e-9;
>> evalfr(G, s) * (s+8)

    0.5625
```

2) Assume X and Y are related by the following transfer function:

$$Y = \left(\frac{8000}{(s+2+j10)(s+2-j10)(s+50)}\right)X$$

a) Use 2nd-order approximations to determine

- The 2% settling time
- The percent overshoot for a step input
- The steady-state output for a step input (x(t) = u(t))

The dominant pole(s) are

$$s = -2 \pm j10 = -10.198\angle \pm 78.69^0 \;\}$$

$$T_s = \left(\frac{4}{2}\right) = 2 \text{ seconds}$$

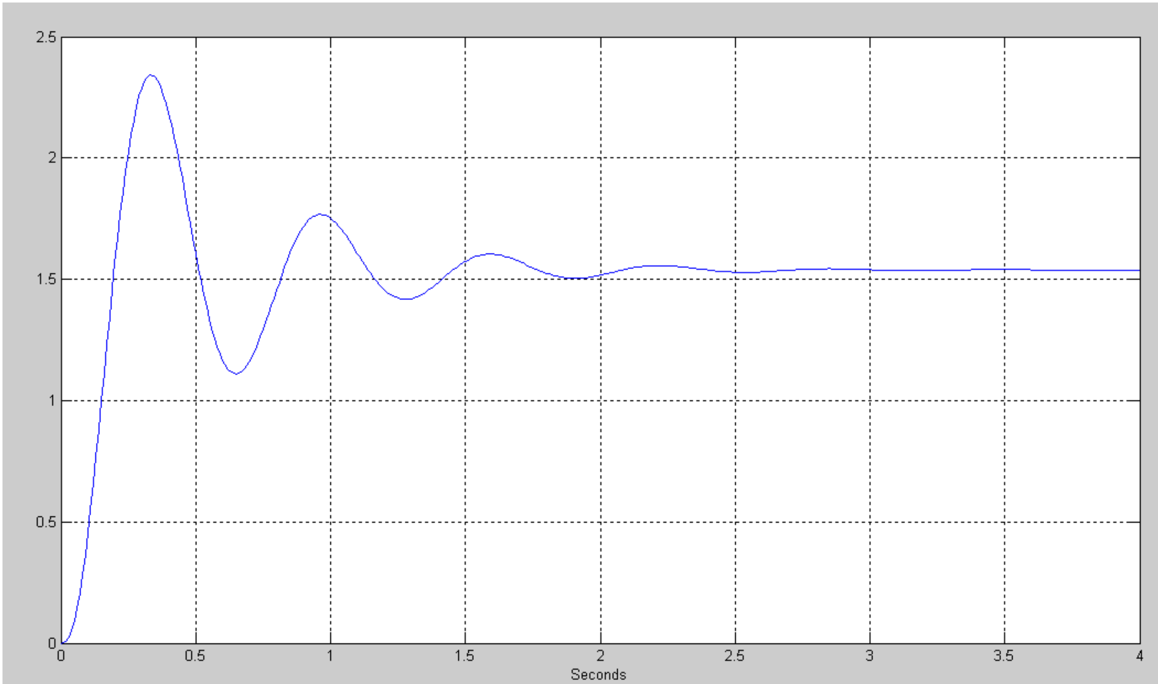
$$\zeta = \cos(78.69^0) = 0.1961$$

$$OS = \exp\left(\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right) = 53.35\%$$

$$DC = \left(\frac{8000}{(s+2+j10)(s+2-j10)(s+50)}\right)_{s=0} = 1.5385$$

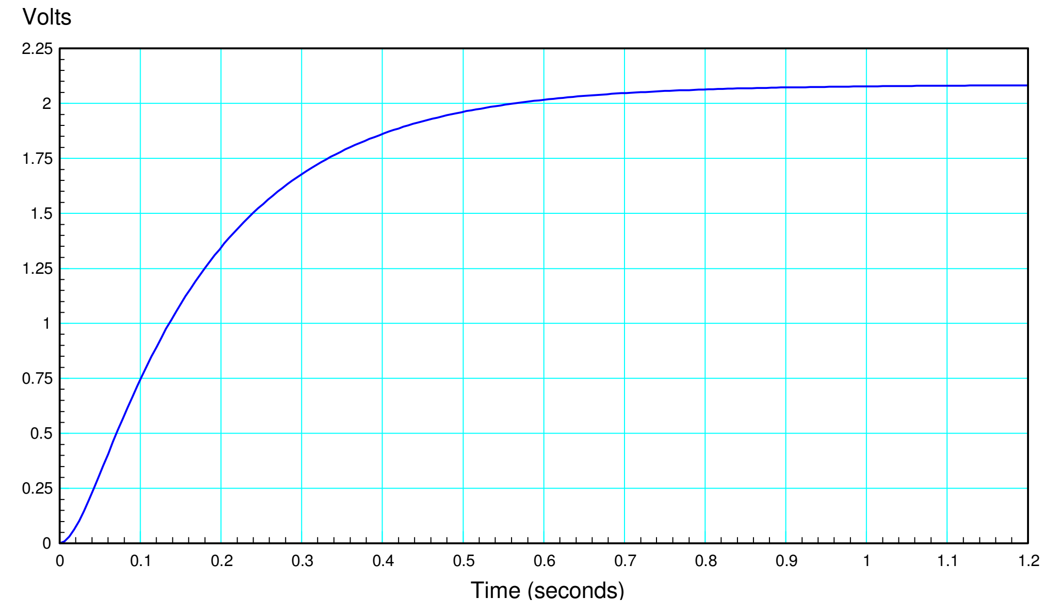
b) Check your answers using the 3rd order model and Matlab, Simulink, of VisSim (your pick)

```
>> G = zpk([],[-2-j*10,-2+j*10,-50],8000)

         8000
-----------------------
(s+50) (s^2 + 4s + 104)

>> t = [0:0.01:4]';
>> y = step(G,t);
>> DC = y(401)

DC =    1.5386

>> OS = (max(y) - DC) / DC

OS =    0.5215

>> plot(t,y)
>> grid
>> xlabel('Seconds')
>>
>> Ts = 0;
>> for i = 1:length(t)
    if( abs( (y(i)-DC)/DC ) > 0.02)
        Ts = t(i);
        end
    end
>> Ts

Ts =    1.9500
```



	Approx	Actual
Ts	2.00 s	1.95 s
% OS	53.35%	52.15%
DC	1.5385	1.5386

3) Determine the transfer function for a system with the following step response:



This is a 1st-order system (no oscillations) meaning the answer is in the form of

$$G(s) = \left(\frac{a}{s+b}\right)$$

To find G(s), we need two pieces of information from the graph.

Settling Time: The 2% settling time is about 0.6 seconds (give of take). That gives

$$b = \frac{4}{0.6} = 6.67$$

DC Gain: The DC gain is about 2.05, meaning

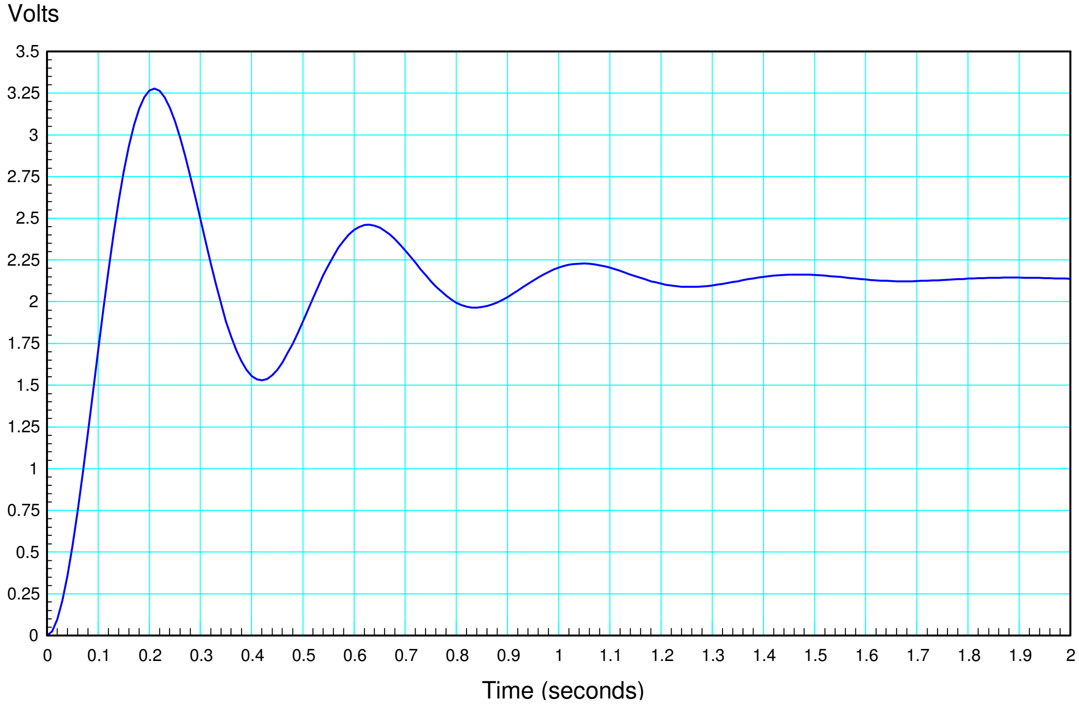
$$\left(\frac{a}{s+b}\right)_{s=0} = 2.05$$

$$a = 2.05b = 13.67$$

or

$$\boxed{G(s) \approx \left(\frac{13.67}{s+6.67}\right)}$$

4) Determine the transfer function for a system with the following step response:



This is a second-order system (oscillations means a complex pole along with its complex conjugate)

$$G(s) = \left(\frac{a}{(s+b+jc)(s+b-jc)}\right)$$

To find G(s), we need three pieces of information from the graph

Frequency of Oscillation: The step response oscillates at c rad/sec

$$c = \left(\frac{3 \text{ cycles}}{1.24 \text{ sec}}\right) 2\pi = 15.20 \frac{rad}{\text{sec}}$$

note: use natural units (rad/sec) rather than icky English units (Hz)

2% settling time: The 2% settling time is about 1.3 seconds

$$b = \left(\frac{4}{1.3}\right) = 3.08$$

DC Gain: The DC gain is about 2.15

$$\left(\frac{a}{(s+b+jc)(s+b-jc)}\right)_{s=0} = 2.15 \quad \Rightarrow \quad a = 517.1$$

$$G(s) \approx \left(\frac{517.1}{(s+3.08+j15.2)(s+3.08-j15.2)}\right)$$