ECE 376 - Homework #11

z-Transforms and Difital Filters - Due Monday, April 28th

LaPlace-Transforms

1) Assume X and Y are related by the following transfer function

$$Y = \left(\frac{2s+7}{s^2+3s+15}\right)X$$

a) What is the differential equation relating X and Y?

Cross multiply

$$(s^2+3s+15)Y = (2s+7)X$$

'sy' means 'the derivative of Y'

$$\boxed{\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 15y = 2\frac{dx}{dt} + 7x}$$

You can also write this using prime-notation (each prime means a derivative)

$$\boxed{y'' + 3y' + 15y = 2x' + 7x}$$

b) Find y(t) assuming

$$x(t) = 2 + 3\sin(4t)$$

Treat this as two separate problems:

$$x_1(t) = 2$$

$$s = 0$$

$$X_1 = 2$$

$$Y_1 = \left(\frac{2s+7}{s^2+3s+15}\right)_{s=0} \cdot (2) = 0.9333$$

$$y_1(t) = 0.9333$$

$$x_2(t) = 3\sin(4t)$$

$$s = j4$$

$$X = 0 - j3$$

$$Y_2 = \left(\frac{2s+7}{s^2+3s+15}\right)_{s=j4} \cdot (0-j3) = -1.9034 - j1.8414$$

$$y_2(t) = -1.9034\cos(4t) + 1.8414\sin(4t)$$

$$\boxed{y(t) = y_1 + y_2 = 0.9333 - 1.9434\cos(4t) + 1.8414\sin(4t)}$$

z-Transforms

2) Assume X and Y are related by the following transfer function

$$Y = \left(\frac{0.25(z+0.5)}{(z-0.8)(z-0.7)}\right)X$$

a) What is the difference equation relating X and Y?

Multiply out and cross multiply

$$(z-0.8)(z-0.7)Y = 0.25(z+0.5)X$$

$$(z^2-1.5z+0.56)Y = 0.25(z+0.5)X$$

Note that 'zY' means 'the next value of Y'

$$\boxed{y(k+2)-1.5y(k+1)+0.56y(k) = 0.25(x(k+1)+0.5x(k))}$$

If you don't like dealing with future values, do a change of variables (or a time shift) to write as

$$\boxed{y(k)-1.5y(k-1)+0.56y(k-2) = 0.25(x(k-1)+0.5x(k-2))}$$

Both answers are correct

b) Find y(t) assuming a sampling rate of T = 0.01 second

$$x(t) = 2+3\sin(4t)$$

Use superposition treating this as two separate problems

$x_1(t) = 2$

$$s = 0$$

$$z = e^{sT} = 1$$

$$X_1 = 2$$

$$Y_1 = \left(\frac{0.25(z+0.5)}{(z-0.8)(z-0.7)}\right)_{z=1} \cdot (2)$$

$$Y_1 = 12.50$$

$$y_1(t) = 12.50$$

$$x_2(t) = 3\sin(4t)$$

$$s = j4$$

$$z = e^{sT} = e^{j0.04}$$

$$X_2 = 0 - j3$$

$$Y_2 = \left(\frac{0.25(z+0.5)}{(z-0.8)(z-0.7)}\right)_{z=e^{j0.04}} \cdot (0 - j3)$$

$$Y_2 = -5.4954 - j17.4973$$

$$y_2(t) = -5.4954\cos(4t) + 17.4973\sin(4t)$$

$$y(t) = y_1(t) + y_2(t)$$

$$\boxed{y(t) = 12.50 - 5.4954\cos(4t) + 17.4973\sin(4t)}$$

c) Find y(t) assuming

$$x(t) = 3u(t)$$

Use z-transforms

$$Y = \left(\frac{0.25(z+0.5)}{(z-0.8)(z-0.7)}\right)\left(\frac{3z}{z-1}\right)$$

Pull out a z and do a partial fraction expansion

$$Y = \left(\frac{0.75(z+0.5)}{(z-1)(z-0.8)(z-0.7)}\right)z$$

$$Y = \left(\left(\frac{18.75}{z-1}\right) + \left(\frac{-48.75}{z-0.8}\right) + \left(\frac{30}{z-0.7}\right)\right)z$$

$$Y = \left(\frac{18.75z}{z-1}\right) + \left(\frac{-48.75z}{z-0.8}\right) + \left(\frac{30z}{z-0.7}\right)$$

Take the inverse z-transform

$$\boxed{y(k) = \left(18.75 - 48.75(0.8)^k + 30(0.7)^k\right)u(k)}$$

Filters in the z-Plane

3) Assume G(s) is a low-pass filter with real poles:

$$G(s) = \left(\frac{500}{(s+1)(s+6)(s+12)}\right)$$

Design a digital filter, G(z), which has approximately the same gain vs. frequency as G(s). Assume a sampling rate of T = 0.01 second.

Plot the gain vs. frequency for both filters from 0 to 50 rad/sec.

Use the mapping from the s-plane to the z-plane of

$$z = e^{sT}$$

$$s = -1 \qquad z = e^{sT} = 0.9900$$

$$s = -6 \qquad z = e^{sT} = 0.9418$$

$$s = -12 \qquad z = e^{sT} = 0.8869$$

So G(z) is of the form

$$G(z) = \left(\frac{k}{(z-0.99)(z-0.9418)(z-0.8869)}\right)$$

Pick k to match the DC gain

$$DC = \left(\frac{500}{(s+1)(s+6)(s+12)}\right)_{s=0} = 6.9444$$

$$DC = \left(\frac{k}{(z-0.99)(z-0.9418)(z-0.8869)}\right)_{z=1} = 6.9444$$

$$k = 0.0004076$$

so

$$\boxed{G(z) = \left(\frac{0.0004076}{(z-0.99)(z-0.9418)(z-0.8869)}\right)}$$

Note: G(z) has extra delay relative to G(s). You can remove this delay by adding z-terms in the numerator to remove this delay. Another valid representation for G(z) would be

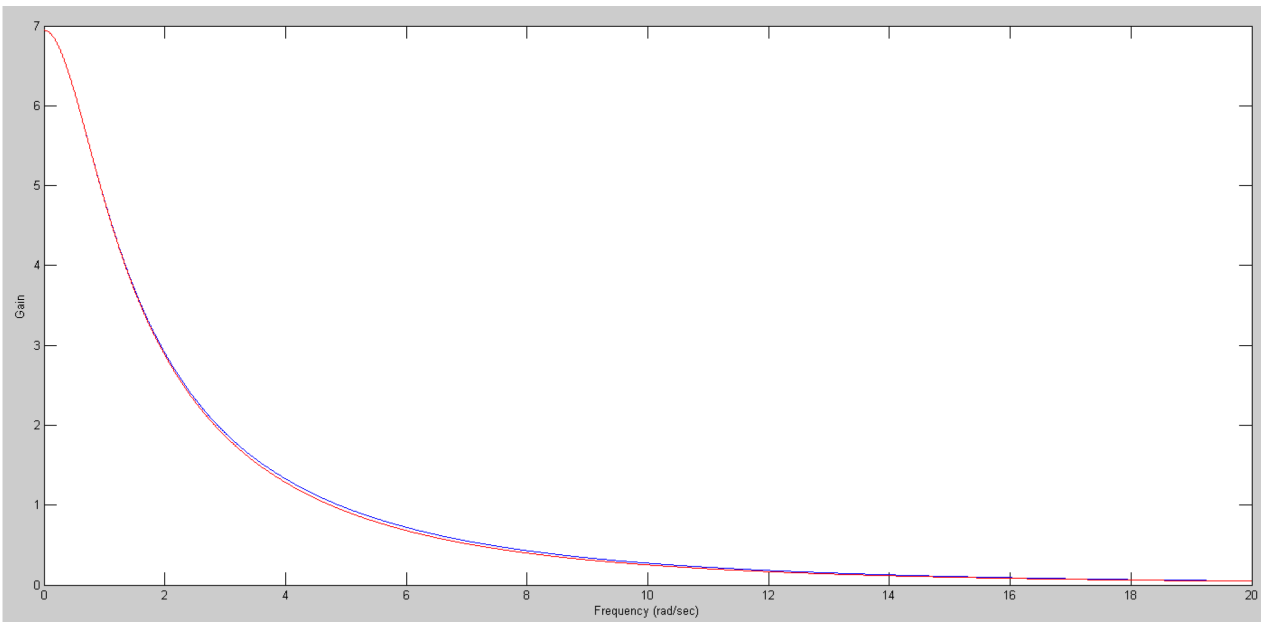
$$G(z) = \left(\frac{0.0004076z^2}{(z-0.99)(z-0.9418)(z-0.8869)}\right)$$

The extra z-terms in the numberator reduce the delay of G(z) but have no impact on the frequency response.

Plotting the gain vs. frequency in Matlab

```
>> w = [0:0.01:20]';
>> s = j*w;
>> Gs = 500 ./ ( (s+1).*(s+6).*(s+12) );

>> T = 0.01;
>> z = exp(s*T);
>> Gz = 0.0004076 ./ ( (z-0.99).*(z-0.9481).*(z-0.8869) );
>> plot(w,abs(Gs),'b',w,abs(Gz),'r')
>> xlabel('Frequency (rad/sec)');
>> ylabel('Gain')
```

The two filters have the same gain vs. frequency. They're the same filter



G(s) (blue) and G(z) (red)

4) Assume G(s) is the following band-pass filter:

$$G(s) = \left(\frac{500}{(s+10)(s^2+2s+400)}\right)$$

Design a digital filter, G(z), which has approximately the same gain vs. frequency as G(s). Assume a sampling rate of T = 0.01 second.

Plot the gain vs. frequency for both filters from 0 to 50 rad/sec.

Same as before: Map from the s-plane to the z-plane as z = exp(sT)

$$s = -10 \qquad z = e^{sT} = 0.9048$$

$$s = -1 + j19.9750 \qquad z = e^{sT} = 0.9704 + j0.1964$$

$$s = -1 - j19.9750 \qquad z = e^{sT} = 0.9704 - j0.1964$$

So, G(z) is of the form

$$G(z) = \left(\frac{k}{(z-0.9048)(z-0.9704+j0.1964)(z-0.9704-j0.1964)}\right)$$

Pick k to match the gain at some frequecy. Pick DC

$$DC = \left(\frac{500}{(s+10)(s^2+2s+400)}\right)_{s=0} = 0.125$$

$$DC = \left(\frac{k}{(z-0.9048)(z-0.9704+j0.1964)(z-0.9704-j0.1964)}\right)_{z=1} = 0.125$$

$$k = 0.0004695$$

so

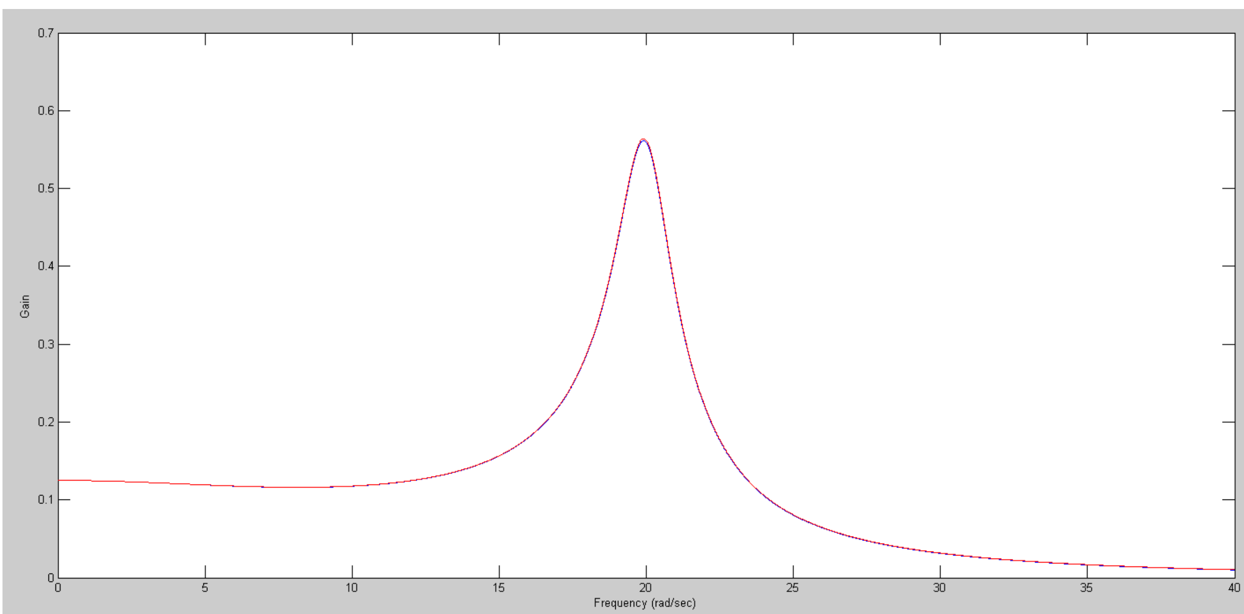
$$G(z) = \left(\frac{0.0004695}{(z-0.9048)(z-0.9704+j0.1964)(z-0.9704-j0.1964)}\right)$$

You can also write this as

$$G(z) = \left(\frac{0.0004695}{(z-0.9048)\left(z^2-1.9407z+0.9802\right)}\right)$$

Plotting the frequency response of G(s) and G(z)

```
>> T = 0.01;
>> s1 = -10;
>> s2 = -1+19.9750i;
>> s3 = conj(s2);

>> z1 = exp(s1*T);
>> z2 = exp(s2*T);
>> z3 = exp(s3*T);

>> s = 0;
>> DCs = 500 / ( (s-s1)*(s-s2)*(s-s3) )
DCs =    0.1250

>> z = 1;
>> DCz = 1 / ( (z-z1)*(z-z2)*(z-z3) );
>> k = DCs / DCz;
>> k = abs(k)
k =  4.6952e-004

>> w = [0:0.01:40]';
>> s = j*w;
>> Gs = 500 ./ ( (s-s1).*(s-s2).*(s-s3) );

>> T = 0.01;
>> z = exp(s*T);
>> Gz = k ./ ( (z-z1).*(z-z2).*(z-z3) );
>> plot(w,abs(Gs),'b',w,abs(Gz),'r')
>> xlabel('Frequency (rad/sec)');
>> ylabel('Gain')
```



Gain of G(s) (blue) and G(z) (red)

The gain is essentially the same: the two filters are equivalent

5) Write a C program to implement the digital filter, G(z)

$$Y = \left(\frac{0.0004695}{(z-0.9048)\left(z^2-1.9407z+0.9802\right)} \right) X$$

Multiply out the denominator

```
>> poly([z1,z2,z3])

ans =     1.0000    -2.8456     2.7362    -0.8869
```

$$Y = \left(\frac{0.0004695}{z^3-2.8456z^2+2.7362z-0.8869} \right) X$$

Cross multiply

$$(z^3 - 2.8456z^2 + 2.7362z - 0.8869)Y = 0.0004695X$$

Convert to a difference equations

```
y(k+3)-2.8456y(k+2)+2.7362y(k+1)-0.8869y(k)=0.0004695x(k)
```

Shift by three (do a change of variable)

```
y(k)-2.8456y(k-1)+2.7362y(k-2)-0.8869y(k-3)=0.0004695x(k-3)
```

Solve for y(k)

```
y(k) = 2.8456y(k-1)-2.7362y(k-2)+0.8869y(k-3)+0.0004695x(k-3)
```

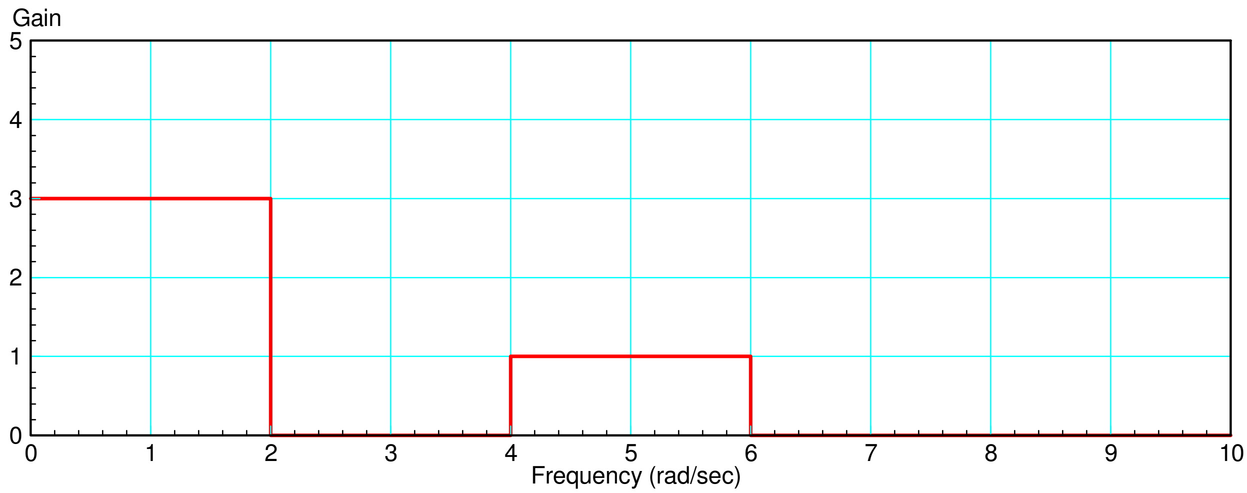
That's your program

```
while(1) {

   x3 = x2;
   x2 = x1;
   x1 = x0;
   x0 = A2D_Read(0);

   y3 = y2;
   y2 = y1;
   y1 = y0;
   y0 = 2.8456*y1 - 2.7362*y2 + 0.8869*y3 + 0.0004695*x3;

   D2A(y0);

   Wait_10ms();
   }
```

FIR Filters

6) Find the impulse response of a filter with the following gain vs. frequency:

- hint: Approximate the waveform by adding up ideal low-pass filters



Note that the impulse response of an ideal low pass filter with a DC gain of 1.00 and a corner at 'a' rad/sec is

$$LPF(a) = \frac{1}{2\pi}\left(\frac{\sin(at)}{t}\right)$$

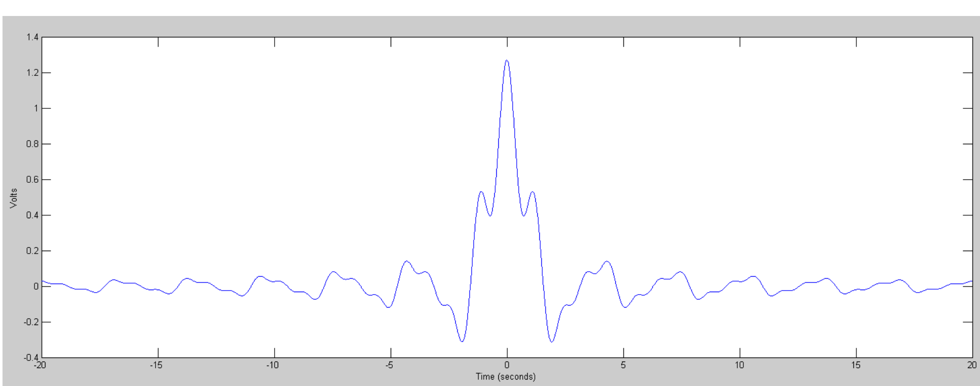
Implement this using step functions

$$3LPF(2) + LPF(6) - LPF(4)$$

$$G(j\omega) = 3\left(\frac{1}{2\pi}\right)\left(\frac{\sin(2t)}{t}\right) + \left(\frac{1}{2\pi}\right)\left(\frac{\sin(6t)}{t}\right) - \left(\frac{1}{2\pi}\right)\left(\frac{\sin(4t)}{t}\right)$$

The impulse response looks like this:

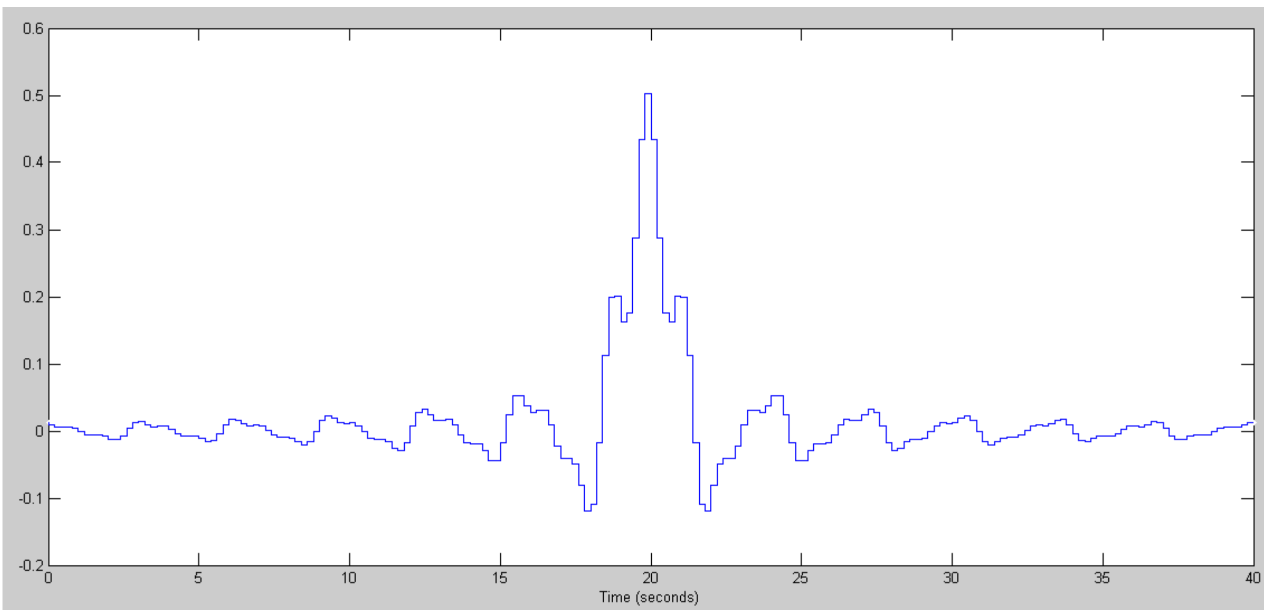
```
>> t = [-20:0.01:20]' + 1e-6;
>> G = 3*sin(2*t)./(t) + sin(6*t)./t - sin(4*t)./t;
>> G = G / (2*pi);
>> plot(t,G)
>> xlabel('Time (seconds)');
>> ylabel('Volts')
```



7) Design a FIR filter to approximate this impulse reaponse. Include in your design

- The sampling rate: 200ms (gives 200 points to represent g(t))
- The length of the window: 40 seconds
- The impulse response of your FIR fitler
- Delay = 20 seconds.

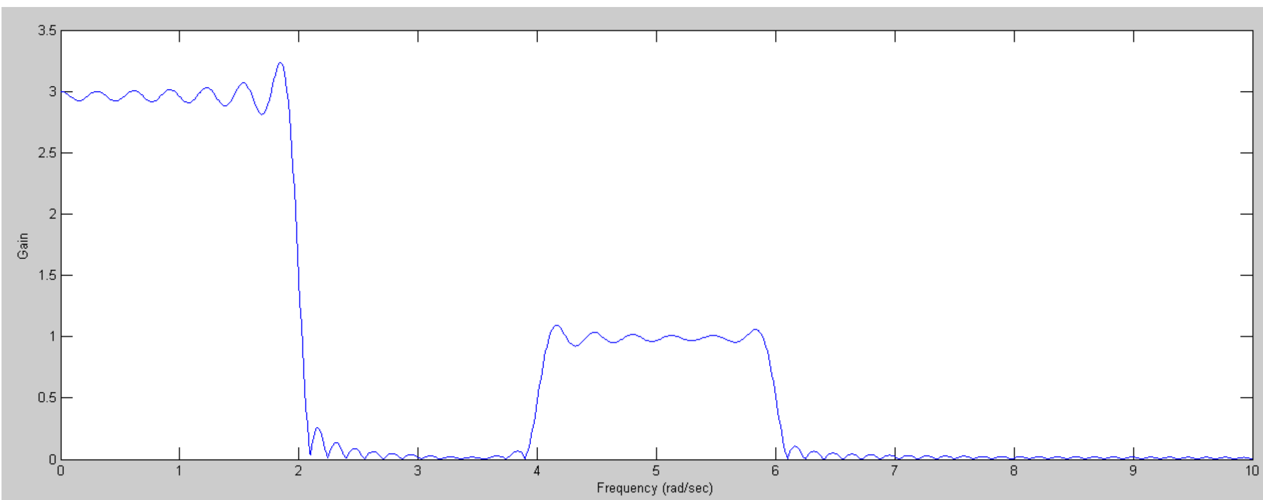
```
>> T = 0.2;
>> t = [-20:T:20]' + 1e-6;
>> H = 3*sin(2*t)./(t) + sin(6*t)./t - sin(4*t)./t;
>> DC = sum(H);
>> H = H * 3 / DC;
>> plot(t+20,H);
>> xlim([0,40])
>> xlabel('Time (seconds)');
>> ylabel('Volts')
```



Impuse Response of FIR filter

8) Plot the gain vs. frequency of your filter

```
T = 0.2;
t = [-20:T:20]' + 1e-6;
H = 3*sin(2*t)./(t) + sin(6*t)./t - sin(4*t)./t;
DC = sum(H);
H = H * 3/DC;

w = [0:0.01:10]';
s = j*w;
z = exp(s*T);

Gw = 0*w;
for i=1:length(t)
    Gw = Gw + H(i) * z.^(1-i);
end

plot(w,abs(Gw))
```



Frequency Response of FIR filter