ECE 376 - Homework #11

z-Transforms and Difital Filters - Due Monday, December 2nd

LaPlace-Transforms

1) Assume X and Y are related by the following transfer function

$$Y = \left(\frac{6s+3}{s^2+7s+18}\right)X$$

a) What is the differential equation relating X and Y?

Cross multiply

$$(s^2+7s+18)Y = (6s+3)X$$

'sY' means *the derivative of y*

$$\boxed{y'' + 7y' + 18y = 6x' + 3x}$$

b) Find y(t) assuming

$$x(t) = 2 + 3\sin(4t)$$

This is a phasor problem along with superposition. Treat this as two separate problems

DC: x(2) = 2

$$s = 0$$

$$X = 2$$

$$Y = \left(\frac{6s+3}{s^2+7s+18}\right)_{s=0} \cdot (2) = 0.3333$$

meaning

$$y(t) = 0.3333$$

AC: x(t) = 3 sin(4t)

$$s = j4$$

$$X = 0 - j3 \qquad\qquad \textit{real = cosine, -imag = sine}$$

$$Y = \left(\frac{6s+3}{s^2+7s+18} \right)_{s=j4} \cdot (0 - j3)$$

$$Y = -0.1371 - j2.5812$$

meaning

$$y(t) = -0.1371\cos(4t) + 2.5812\sin(4t)$$

The total answer is the DC term plus the AC term (superposition)

$$\boxed{y(t) = 0.3333 - 0.1371\cos(4t) + 2.5812\sin(4t)}$$

Comment: A common mistake students make is they take the two terms:

DC: $Y = 0.3333$

AC: $Y = -0.1371 - j2.5812$

and add them together

DC + AC: $Y = 0.1963 - j2.5812$

You can't do that.

- *The DC term represents y(t) at 0 rad/sec.*
- *The AC term represents y(t) as 4 rad/sec.*

In the total answer (above in the box), the AC and DC terms don't combine: you can't simplify this answer.

z-Transforms

2) Assume X and Y are related by the following transfer function

$$Y = \left(\frac{0.25(z+1)}{(z-0.9)(z-0.4)}\right)X$$

a) What is the difference equation relating X and Y?

Multiply out and cross multiply

$$(z^2 - 1.3x + 0.36)Y = 0.25(z+1)X$$

Note that 'zY' means 'the next value of y(k)' or 'y(k+1')

$$\boxed{y(k+2) - 1.3y(k+1) + 0.36y(k) = 0.25(x(k+1) + x(k))}$$

b) Find y(t) assuming a sampling rate of T = 0.01 second

$$x(t) = 2 + 3\sin(4t)$$

Again, use superposoition

DC: x(t) = 2

$$X = 2$$

$$s = 0$$

$$z = e^{sT} = 1$$

$$Y = \left(\frac{0.25(z+1)}{(z-0.9)(z-0.4)}\right)_{z=1} \cdot (2)$$

$$Y = 16.6667$$

meaning

$$y(t) = 16.6667$$

AC: x(t) = 3 sin(4t)

$$X = 0 - j3$$

$$s = j4$$

$$z = e^{sT} = e^{j0.04}$$

$$Y = \left(\frac{0.25(z+1)}{(z-0.9)(z-0.4)}\right)_{z=e^{j0.04}} \cdot (0 - j3)$$

$$Y = -9.7297 - j21.2246$$

meaning

$$y(t) = -9.2797\cos(4t) + 21.2246\sin(4t)$$

The total answer is DC + AC

$$\boxed{y(t) = 16.6667 - 9.2797\cos(4t) + 21.2246\sin(4t)}$$

c) Find y(t) assuming

$$x(t) = 2u(t)$$

This is a z-transform problem. Replace x(t) with the transform for X(z)

$$Y = \left(\frac{0.25(z+1)}{(z-0.9)(z-0.4)}\right)\left(\frac{z}{z-1}\right)$$

Pull out a z

$$Y = \left(\frac{0.25(z+1)}{(z-1)(z-0.9)(z-0.4)}\right)z$$

Do a patrial fraction expansion

$$Y = \left(\left(\frac{8.3333}{z-1}\right) + \left(\frac{-9.50}{z-0.9}\right) + \left(\frac{1.1667}{z-0.4}\right)\right)z$$

Multiply through by z

$$Y = \left(\frac{8.3333z}{z-1}\right) + \left(\frac{-9.50z}{z-0.9}\right) + \left(\frac{1.1667z}{z-0.4}\right)$$

Take the inverse z-transform

$$\boxed{y(k) = \left(8.3333 - 9.50(0.9)^k + 1.1667(0.4)^k\right)u(k)}$$

Filters in the z-Plane

3) Assume G(s) is a low-pass filter with real poles:

$$G(s) = \left(\frac{500}{(s+4)(s+7)(s+10)}\right)$$

Design a digital filter, G(z), which has approximately the same gain vs. frequency as G(s). Assume a sampling rate of T = 0.01 second.

Plot the gain vs. frequency for both filters from 0 to 50 rad/sec.

Convert from the s-plane to the z-plane as z = exp(sT)

$$s = -4 \qquad z = e^{sT} = 0.9608$$

$$s = -7 \qquad z = e^{sT} = 0.9324$$

$$s = -10 \qquad z = e^{sT} = 0.9048$$

meaning G(z) is of the form:

$$G(z) = \left(\frac{k}{(z-0.9608)(z-0.9324)(z-0.9048)}\right)$$

Pick 'k' to match the DC gain

$$G_s = \left(\frac{500}{(s+4)(s+7)(s+10)}\right)_{s=0} = 1.7857$$

$$G_z = \left(\frac{k}{(z-0.9608)(z-0.9324)(z-0.9048)}\right)_{z=1} = 1.7857$$

$$k = 0.00045049$$

so

$$\boxed{G(z) = \left(\frac{0.00045049}{(z-0.9608)(z-0.9324)(z-0.9048)}\right)}$$

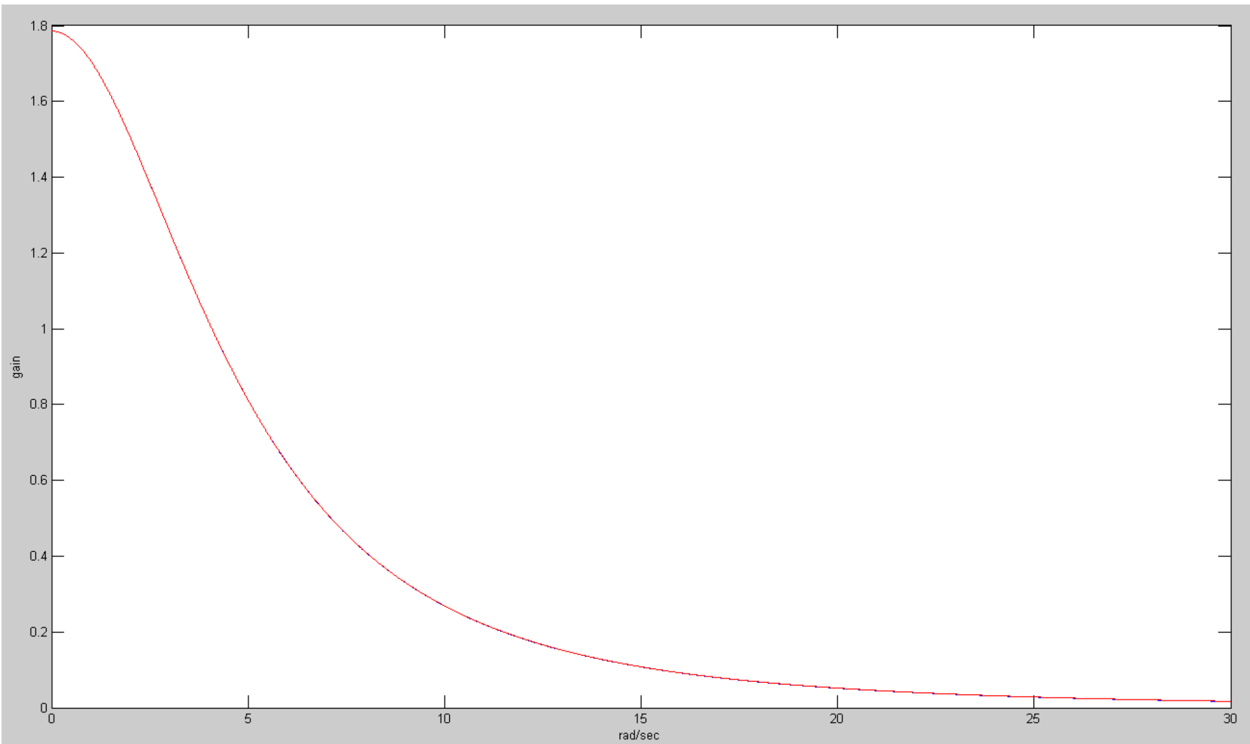
sidelight: The gain of G(z) should match G(s). The phase shift will be a little off, however, due to G(z) having too much delay. If the delay is important to you (such as in ECE 461 Controls). add zeros at z=0 until the phase (delay) matches as close as possible.

Check: Plot the gain vs. frequency for G(s) and G(z). In Matlab:

```
>> w = [0:0.01:30]';
>> s = j*w;
>> T = 0.01;
>> z = exp(s*T);
>> Gs = 500 ./ ( (s+4).*(s+7).*(s+10) );
>> Gz = 0.00045049 ./ ( (z-0.9608).*(z-0.9324).*(z-0.9048) );
>> plot(w,abs(Gs),'b',w,abs(Gz),'r')
>> xlabel('rad/sec')
>> ylabel('gain')
```

The gain vs. frequency of the two filters are almost identical.

- From a user standpoint, I don't really care how you implement this filter: G(s) or G(z)
- The gain vs. the frequency is the same either way



4) Assume G(s) is the following band-pass filter:

$$G(s) = \left(\frac{500}{(s+4)(s^2+4s+100)}\right)$$

Design a digital filter, G(z), which has approximately the same gain vs. frequency as G(s). Assume a sampling rate of T = 0.01 second.

Plot the gain vs. frequency for both filters from 0 to 50 rad/sec.

Same procedure as before:

$$s = -4 \qquad z = e^{sT} = 0.9608$$

$$s = -2 + j9.7980 \qquad z = e^{sT} = 0.9755 + j0.0959$$

$$s = -2 - j9.7980 \qquad z = e^{sT} = 0.9755 - j0.0959$$

meaning

$$G(z) = \left(\frac{k}{(z-0.9608)(z-0.9755+j0.0959)(z-0.0755-j0.0959)}\right)$$

For conveniance, multiply together the complex terms

$$G(z) = \left(\frac{k}{(z-0.9608)\left(z^2-1.951z+0.9608\right)}\right)$$

Pick 'k' to match the DC gain

$$DC = \left(\frac{500}{(s+4)(s^2+4s+100)}\right) = 1.25$$

$$DC = \left(\frac{k}{(z-0.9608)\left(z^2-1.951z+0.9608\right)}\right)_{z=1} = 1.25$$

$$k = 0.00048006$$

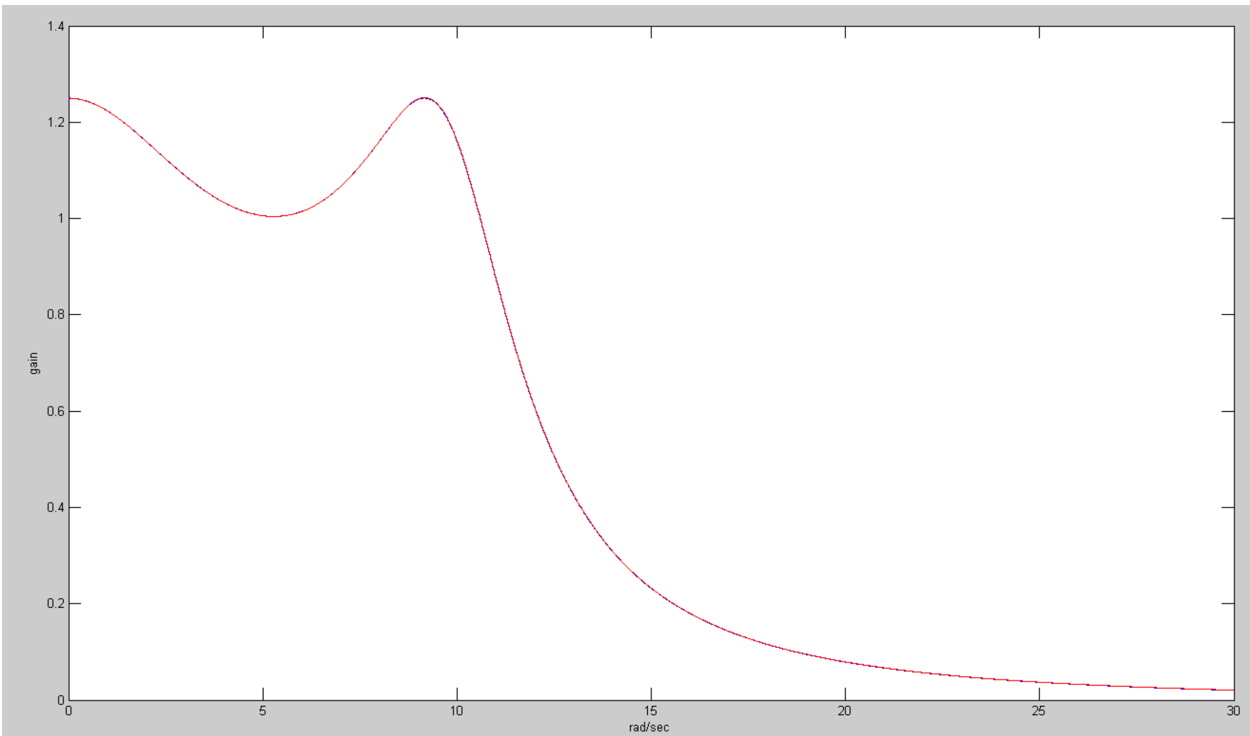
so

$$G(z) = \left(\frac{0.00048006}{(z-0.9608)\left(z^2-1.951z+0.9608\right)}\right)$$

Check: Plot the gain vs. frequency for G(s) and G(z)

```
>> w = [0:0.01:30]';
>> s = j*w;
>> T = 0.01;
>> z = exp(s*T);
>> Gs = 500 ./ ( (s+4).*(s.^2 + 4*s + 100) );
>> Gz = 0.00048006 ./ ( (z-0.9608).*(z.^2 - 1.951*z + 0.9608) );
>> plot(w,abs(Gs),'b',w,abs(Gz),'r')
>> xlabel('rad/sec')
>> ylabel('gain')
```

G(s) and G(z) have the same gain vs. frequency. Essentially, they're the same filter.



5) Write a C program to implement the digital filter, G(z)

$$Y = \left(\frac{0.00048006}{(z-0.9608)\left(z^2-1.951z+0.9608\right)} \right) X$$

There are several ways of doing this. The most straight-forward way (not the best option) is to multuply out the polynomials

$$Y = \left(\frac{0.00048006}{z^3-2.9118z^2+2.8353z-0.9231} \right) X$$

Cross multiply

$$(z^3 - 2.9118z^2 + 2.8353z - 0.9231)Y = (0.00048006)X$$

Write as a difference equations

$$y(k+3) - 2.9118y(k+2) + 2.8353y(k+1) - 0.9231y(k) = 0.00048006x(k)$$

shift by three (change of variable)

$$y(k) - 2.9118y(k-1) + 2.8353y(k-2) - 0.9231y(k-3) = 0.00048006x(k-3)$$

solve for y(k)

$$y(k) = 2.9118y(k-1) - 2.8353y(k-2) + 0.9231y(k-3) + 0.00048006x(k-3)$$

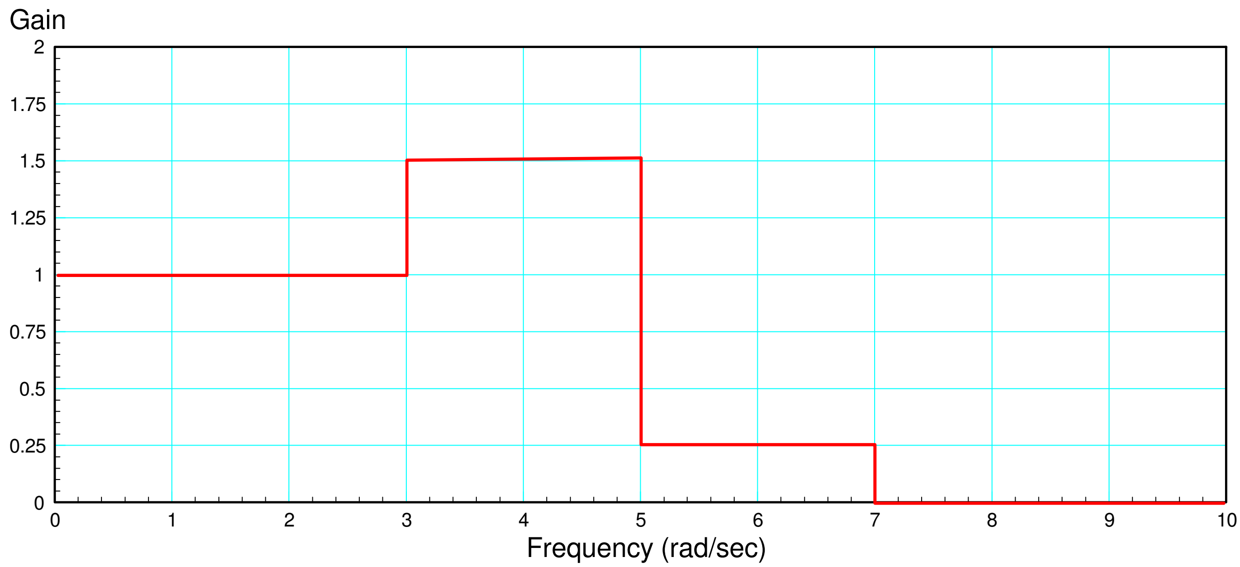
That's pretty much your program

```
while(1) {
   x3 = x2;
   x2 = x1;
   x1 = x0;
   x0 = A2D_Read(0);

   y3 = y2;
   y2 = y1;
   y1 = y0;
   y0 = 2.9118*y1 - 2.8353*y2 + 0.9231*y3 + 0.00048006*x3;
   D2A(y0);
   Wait_ms(10);
   }
```

FIR Filters

6) Find the impulse response of a filter with the following gain vs. frequency:

- hint: Approximate the waveform by adding up ideal low-pass filters



Add rectangles to get this shape

$$h(t) = 0.25 \cdot LPF(7) + 1.25 \cdot LPF(5) - 0.5 \cdot LPF(3)$$

$$h(t) = 0.25\left(\frac{\sin(7t)}{t}\right) + 1.25\left(\frac{\sin(5t)}{t}\right) - 0.5\left(\frac{\sin(3t)}{t}\right)$$

times a fudge factor to make the DC gain equal to 1.0000 (the total area is 1.000)

```
>> dt = 0.001;
>> t = [-30:dt:30]' + 1e-9;
>> h1 = sin(7*t)./t;
>> sum(h1) * dt

ans =     3.1500
```

Apparently, the fudge factor is $\frac{1}{\pi}$ (not too surprised)

$$h(t) = \frac{0.25}{\pi}\left(\frac{\sin(7t)}{t}\right) + \frac{1.25}{\pi}\left(\frac{\sin(5t)}{t}\right) - \frac{0.5}{\pi}\left(\frac{\sin(3t)}{t}\right)$$

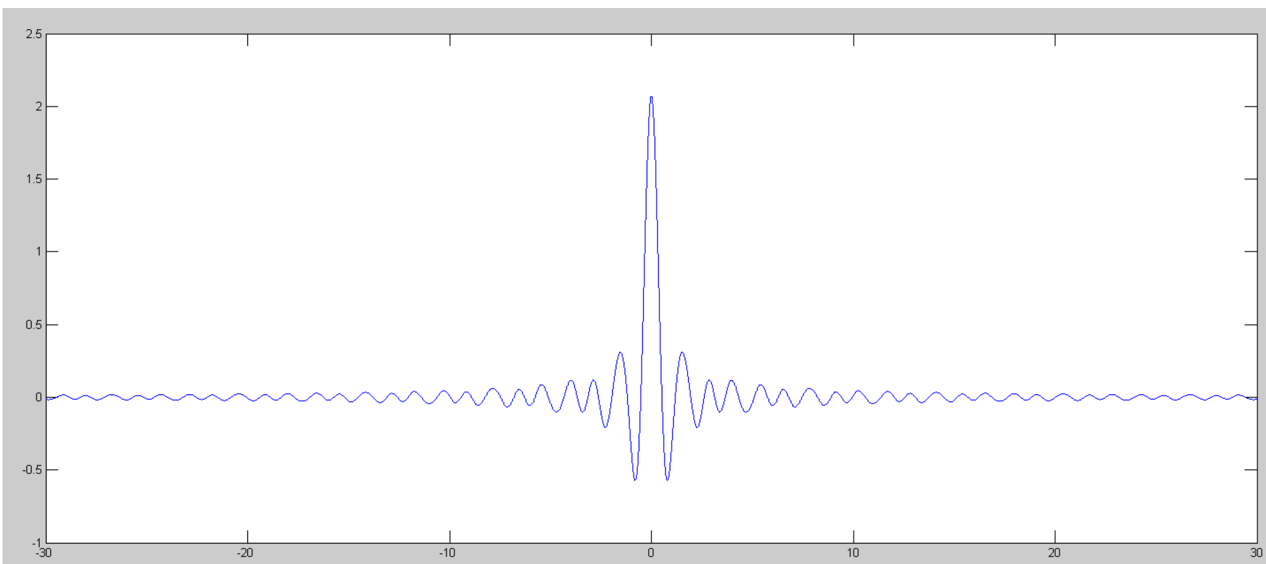
7) Design a FIR filter to approximate this impulse response. Include in your design

- The sampling rate
- The length of the window (10 seconds?)
- The impulse response of your FIR fitler.

Start with looking at the impulse response for h(t). In Matlab:

```
>> dt = 0.001;
>> t = [-30:0.001:30]' + 1e-9;
>> h1 = 0.25/pi * sin(7*t) ./ t;
>> h2 = 1.25/pi * sin(5*t) ./ t;
>> h3 = -0.5/pi * sin(3*t) ./ t;
>> h = h1 + h2 + h3;
>> plot(t,h)
>> xlim([-30,30])
>> DC = sum(h) * dt

DC =  0.9954
```

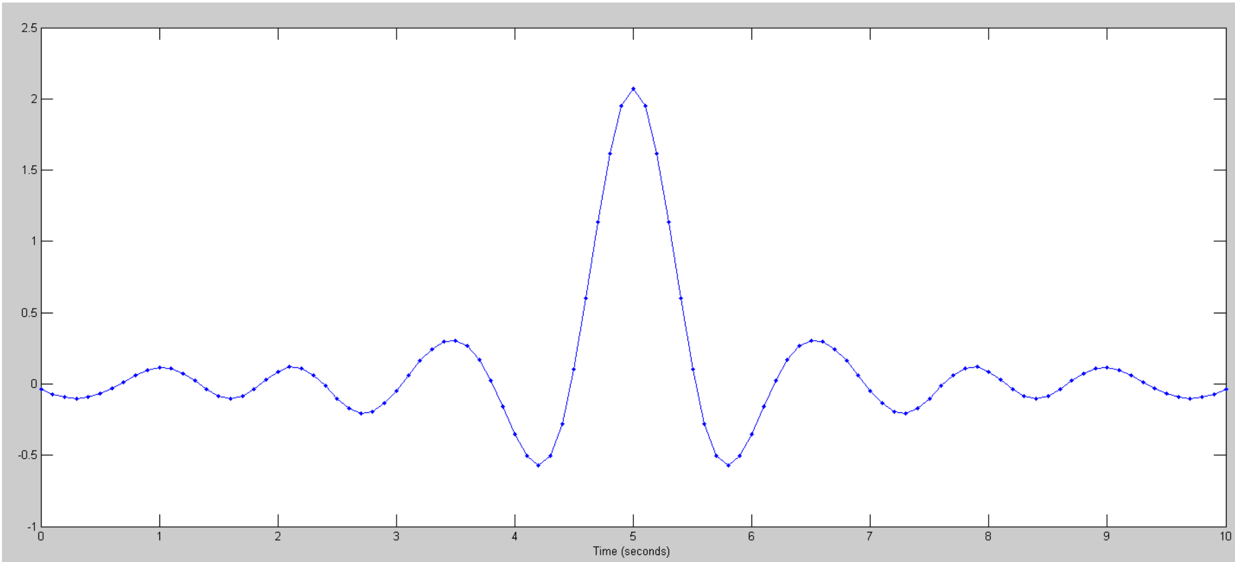
Almost 1.000 - so it looks right.



The impulse reasponse goes from -infinity to +infinity - which is hard to implement. Arbitrarily, truncate at -5 seconds to + 5 seconds and shift right by 5 to make causal:

This is an analog signal, meaning an infinite number of points in the range of (-5s, +5s). Infinity is also a difficult number to program, so approximat this with 100 points:

```
>> dt = 0.1;
>> t = [-5:dt:5]' + 1e-9;
>> h1 = 0.25/pi * sin(7*t) ./ t;
>> h2 = 1.25/pi * sin(5*t) ./ t;
>> h3 = -0.5/pi * sin(3*t) ./ t;
>> h = h1 + h2 + h3;
>> plot(t+5, h)
>> xlim([0,10]);
>> xlabel('Time (seconds)')
```

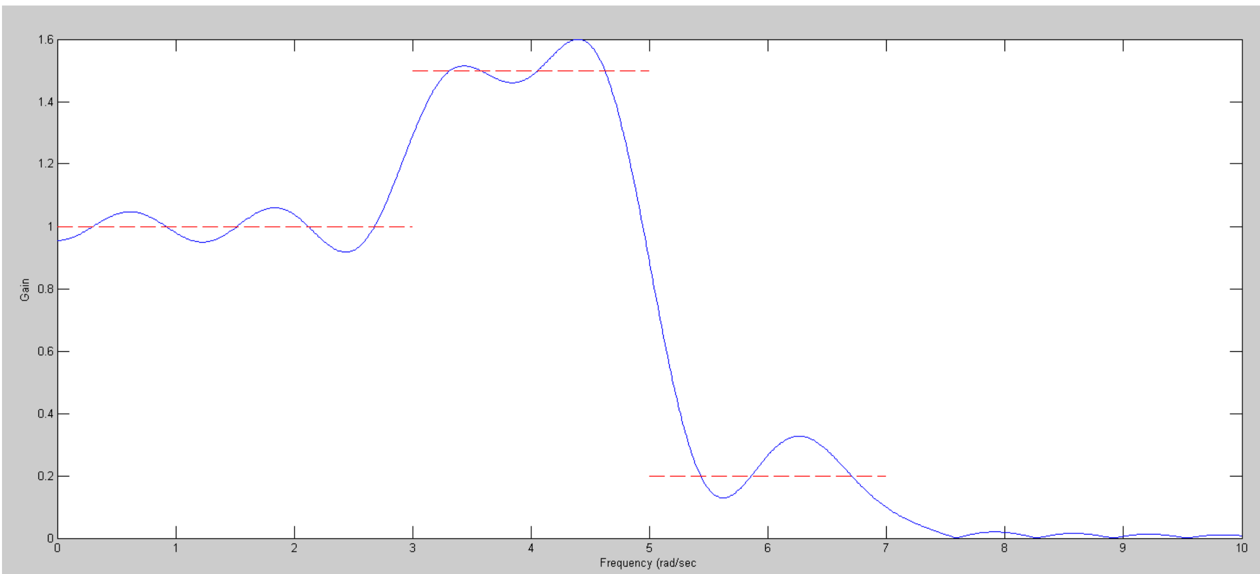


Impulse response of FIR filter

There are many other ways to approximate h(t). The acid test is if it works (problem #8)

8) Plot the gain vs. frequency of your filter

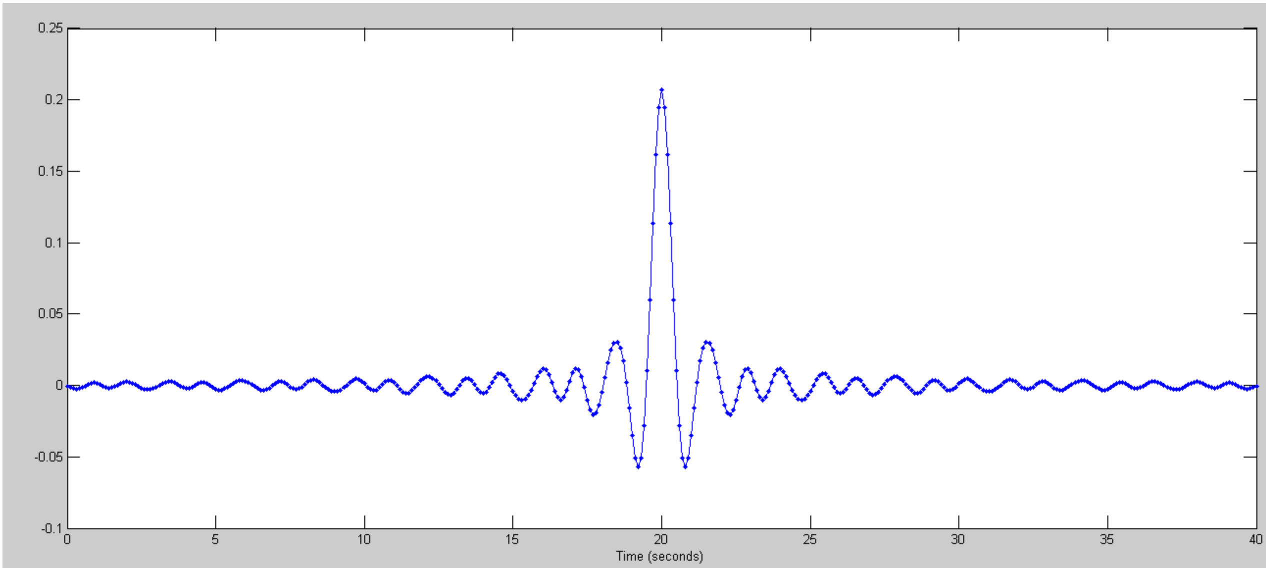
```
>> w = [0:0.01:10]';
>> s = j*w;
>> T = dt;
>> z = exp(s*T);

>> Gw = 0*w;
>> for i = 1:length(h)
     Gw = Gw + h(i) * z .^ (1-i);
     end
>> plot(w,abs(Gw))
>> hold on
>> plot([0,3],[1,1],'r--')
>> plot([3,5],1.5*[1,1],'r--')
>> plot([5,7],0.25*[1,1],'r--')
>> xlabel('Frequency (rad/sec')
>> ylabel('Gain');
>>
```



The approximation isn't great. It would be closer if

- you went further out than 5 seconds, and
- you used more than 100 points in the FIR filter

If you go from -20 to +20 seconds, the impulse response contains 800 points:



and the frequency response is closer: