Markov Chains with Absorbing States

ECE 341: Random Processes Lecture #23

note: All lecture notes, homework sets, and solutions are posted on www.BisonAcademy.com

Absorbing States

Markov chains solve problems of the form

$$x(k+1) = A x(k)$$
 $x(k=0) = X_0$

The previous lecture looked at tossing a ball around

• Four players

An absorbing state is where a player keeps the ball

• State C in this case



State Transistion Matrix with Absorbing States

A diagonal element of '1' indicated an absorbing state





Steate-State Solution

If

- There is only one absorbing state, and
- It is reachable

The ball always ends up at the absorbing state

```
>> A = [0.2,0.1,0,0.2 ; 0.3,0.3,0,0.1 ; 0.1,0.2,1,0.4 ; 0.4,0.4,0,0.3]
```

0.2000	0.1000	0	0.2000
0.3000	0.3000	0	0.1000
0.1000	0.2000	1.0000	0.4000
0.4000	0.4000	0	0.3000

>> A^100

0.0000	0.0000	0	0.0000
0.0000	0.0000	0	0.0000
1.0000	1.0000	1.0000	1.0000
0.0000	0.0000	0	0.0000

Eigenvalues & Eigenvectors

This also shows up with the eigenvalues and eigenvectrors

- The absorbing state has an eigenvalue of 1.0000
 - It never goes away
- It's eigenvector is the absorbing staet
 - State C

Eigenvectors

А	0	0.2023	0.6250	0.6250
В	0	0.2390	-0.5359 - 0.2	2959i -0.5359 + 0.2959i
С	1.0000	-0.8549	0.1538 - 0.0	0.1538 + 0.0847i
D	0	0.4135	-0.2429 + 0.3	8806i -0.2429 - 0.3806i

Eigenvalues

1.0000 0.7269	0.0365 - 0.0745i	0.0365 + 0.0745i
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The z-transform for the absorbing state has a pole at z=1

• A starts with the ball

The z-transform for each state is:

$$A(z) = \left(\frac{(z-0.1)(z-0.5)z}{(z-0.7269)(z^2-0.07308z+0.006878)}\right)$$

$$B(z) = \left(\frac{0.3z(z-0.1667)}{(z-0.7269)(z^2-0.07308z+0.006878)}\right)$$

$$C(z) = \left(\frac{0.1z(z+1.631)(z-0.03066)}{(z-1)(z-0.7269)(z^2-0.07308z+0.006878)}\right) \quad <<< \text{ pole at } z = 1$$

$$D(z) = \left(\frac{0.4z^2}{(z-0.7269)(z^2-0.07308z+0.006878)}\right)$$

note:

- Only state C has a pole at z=1
- C is an absorbing state

As you run	a simulati	on, you wir	nd up at the	e absorbing s	state
k	p(A)	р(В)	р(С)	p(D)	
0	0.2500	0.2500	0.2500	0.2500	
1	0.1250	0.1750	0.4250	0.2750	
2	0.0975	0.1175	0.5825	0.2025	
3	0.0718	0.0847	0.6967	0.1468	
4	0.0522	0.0616	0.7796	0.1066	
5	0.0379	0.0448	0.8398	0.0775	
6	0.0276	0.0326	0.8835	0.0563	
7	0.0200	0.0237	0.9153	0.0410	
8	0.0146	0.0172	0.9385	0.0298	
9	0.0106	0.0125	0.9553	0.0216	
10	0.0077	0.0091	0.9675	0.0157	
11	0.0056	0.0066	0.9764	0.0114	
12	0.0041	0.0048	0.9828	0.0083	
13	0.0030	0.0035	0.9875	0.0060	
14	0.0021	0.0025	0.9909	0.0044	
15	0.0016	0.0018	0.9934	0.0032	
16	0.0011	0.0013	0.9952	0.0023	
17	0.0008	0.0010	0.9965	0.0017	
18	0.0006	0.0007	0.9975	0.0012	
19	0.0004	0.0005	0.9982	0.0009	
20	0.0003	0.0004	0.9987	0.0006	

There can be several absorbing states

Example: Win By Two game

• Example: Tennis

Absorbing States

- A is up 2 games
 - A wins
- B is up 2 games
 - B wins



Win Two vs Win By Two

There's a big difference between

- First to win 2 games and
- First to win by 2 games

The former is a binomial distribution

• 3 game series

The latter is a Markov chain

- Series can go on forever
- Example: 2022 US Open
 - Alcaraz vs. Sinner
 - 5 hour 15 minute tennis match



Example: First to Win 2 Games

What is the chance A wins a 3-game series?

- A has a 70% chance of winning any given game
- Example: NBA semi-finals

Solution: A can win three ways (enumeration):

- AA
- ABA
- BAA

The total odds of A winning is

$$p(A) = p^2 + 2p^2q$$
$$p(A) = 0.7840$$



Example: First to Win By 2 Games

There are five states:

- A is up 2 games (A wins)
- A is up 1 game
- Tied
- B is up 1 game
- B is up 2 games (B wins)



Win By 2 Games: State Transition Matrix

- 70% chance of a win
- 0% chance of a tie
- 30% chance of a loss

$$X(k+1) = \begin{bmatrix} 1.0 \ 0.7 \ 0 \ 0 \ 0 \\ 0 \ 0.3 \ 0 \ 0.7 \ 0 \\ 0 \ 0 \ 0.3 \ 0 \ 0.7 \ 0 \\ 0 \ 0 \ 0.3 \ 1.0 \end{bmatrix} X(k)$$



Win-By-2 Series: Who will Win?

Find the steady-state solution

$$x(k+1) = Ax(k) = x(k)$$
$$(A - I)x(k) = 0$$

This doesn't help in this case due to the absorbing states. If you try to solve, you get

$$\begin{bmatrix} 0 \ 0.7 \ 0 \ 0 \ 0 \\ 0 \ -1 \ 0.7 \ 0 \ 0 \\ 0 \ 0.3 \ -1 \ 0.7 \ 0 \\ 0 \ 0 \ 0.3 \ -1 \ 0 \\ 0 \ 0 \\ 0 \ 0 \ 0.3 \ 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} = 0 \qquad X(\infty) = \begin{bmatrix} a \\ 0 \\ 0 \\ 0 \\ e \end{bmatrix}$$

result: someone wins

Option 2: Eigenvectors.

The eigenvalues and eigenvectors tell you

- How the system behaves (eigenvalues) and
- What behaves that way.

```
[M,V] = eiq(A)
   1.0000
                      -0.8033
                                 0.2846
                                         -0.5384
                  0
                  0
                       0.4039
                               -0.6701
                                        0.7692
         0
                       0.3739
                                       -0.0000
        0
                  0
                               0.6203
        0
                    0.1731 - 0.2872
                                        -0.3297
                  0
             1.0000
                    -0.1476
                              0.0523
                                          0.0989
        0
             1.0000
V:
   1.0000
                       0.6481
                                -0.6481
                                                0
```

The eigenvectors tell you that eventually,

- A wins (first eigenvector), or
- B wins (second eivenvector).

Option 3: Play the game a large number of times (100 times). In Matlab:

A = [1, 0, 0, 0, 0; 0.7, 0, 0.3, 0, 0; 0, 0.7, 0, 0.3, 0; 0, 0, 0.7, 0, 0.3; 0, 0, 0, 0, 1]'

1.0000	0.7000	0	0	0
0	0	0.7000	0	0
0	0.3000	0	0.7000	0
0	0	0.3000	0	0
0	0	0	0.3000	1.0000
A^100				
1.0000	0.9534	0.8448	0.5914	0
0	0	0.0000	0	0
0	0.0000	0	0.0000	0
0	0	0.0000	0	0
0	0.0466	0.1552	0.4086	1.0000

A^100 tells you the probability of

- A winning (first row) and
- B winning (last row)

The columns tell you the probability if you offer odds:

- Column 1: Player A starts with a +2 game advantage (player A always wins)
- Column2: Player A starts with a +1 game advantage (A wins 95.34% of the time)
- Column 3: Player A starts with a +0 game advantage (A wins 84.48% of the time)
- Column 4: Player B starts with a +1 game advantage (A wins 59.14% of the time)
- Column 5: Player B starts with a +2 game advantage (B always wins)

Odds +2	+1	+0	-1	-2	
1.0000	0.9534	0.8448	0.5914	0	A wins
0	0	0.0000	0	0	
0	0.0000	0	0.0000	0	
0	0	0.0000	0	0	
0	0.0466	0.1552	0.4086	1.0000	B wins

Good Money After Bad

- If you start losing, keep gambling to recoup your losses
- This tends to result in you getting further behind
- You're risking money you have (good money) to recoup money you lost (bad money)



Good Money After Bad

For example, suppose you play a game of chance.

- 55% of the time you win and earn \$1.
- 45% of the time you lose and lose \$1.

Keep playing until you are up \$10

• Absorbing state

In theory, you always up \$10

- Only one absorbing state
- Sounds like a safe way to make \$10

• Keep playing until you are up \$10. • After 200 games, you're almost always up \$10 % game of chance X = zeros(200, 5);for k=1:5Bank = 0;for n=2:200; if(Bank < 10)if (rand < 0.55) Bank = Bank + 1;else Bank = Bank - 1;end end X(n,k) = Bank;end end k = [1:200]';plot(k,X)

Monte-Carlo Simulation



Change the problem

• p(winning) = 45%

There still is only one absorbing state

- Quit when you're up \$10
- In theory, you *always* end at the absorbing state

Sometimes you win

• Reach the absorbing state

Other times the game keeps going

- What's happening?
- Why don't you always end at the absorbing state?



Better Model

Add a second absorbing state:

- Quit when up \$10
- Quit when down \$100
 - The house no longer accepts your money.

Monte-Carlo Simulation: In 10,000 games

- 6,647 times you're up \$10
- 3,353 times you're down \$100

If you're losing, walk away.

- It could be you're losing because you're just not that good
- Your odds of winning are actually 45%, not 55%



z-Transforms & Markov Chains

Problem: Determine the probability of player A winning after k games.

- New topic
- This is where z-transforms shine.

Solution: Express the system in state-space form:

```
X(k+1) = AX(k) + BU(k)Y = CX(k)
```

If U(k) is an impulse function, you get the impulse response:

$$zX = AX + B$$
$$Y = CX$$

This also works for Markov chains where

- B is the initial condition: X(0)
- C tells you which state you want to look at.

For example, for the problem of winning by 2 games,

A = 1.0000 0.7000 0 0 0.7000 0 0 $\left(\right)$ 0.3000 0.7000 0 0 0.3000 0 0 0 0 0.3000 1.0000 0 0 X0 = [0;0;1;0;0]0 0 1 0 0 C = [1, 0, 0, 0, 0] % A wins 1 0 0 C = 0 0

0

0

0

 $\left(\right)$

You can now find the Y(z) (or the impulse response)

• Multiply by z (as per last lecture)

```
G = ss(A, X0, C, 0, 1);
tf(G)
0.49 z
z^{4} - z^{3} - 0.42 z^{2} + 0.42 z + 1.821e-018
sampling time (seconds): 1
zpk(G)
0.49 z
z (z-1) (z-0.6481) (z+0.6481)
```

Sampling time (seconds): 1

This tells you that

$$Y(z) = \left(\frac{0.49z}{(z-1)(z-0.6481)(z+0.6481)}\right)$$

The time response is from the *impulse* function

y = impulse(G)



You can also find the explicit function for y(k) using z-transfoms.

 $Y(z) = \left(\frac{0.49z}{(z-1)(z-0.6481)(z+0.6481)}\right)$

Pull ou a z and do partial fractions

$$Y = \left(\left(\frac{0.8449}{z - 1} \right) + \left(\frac{-1.0742}{z - 0.6481} \right) + \left(\frac{0.2294}{z + 0.6481} \right) \right) z$$

Take the inverse z-transform

$$y(k) = \left(0.8449 - 1.0742 \ (0.6481)^k + 0.2294 \ (-0.6481)^k\right) \ u(k)$$



Summary

An absorbing state is state you never leave once you enter that state

• Example: Match is over once you're up 2 games

If there is only one absorbing state and it's reachable, you always end up at the absorbing state

• z-Transforms are helpful in computing the path to that state

If there are two absorbing states, you can find the probability of ending at each absorbing state by

- Matrix multiplication,
- Monte-Carlo simulations, or
- z-Transforms