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# **Gamma and Poisson Distribution**

**ECE 341: Random Processes  
Lecture #16**

note: All lecture notes, homework sets, and solutions are posted on [www.BisonAcademy.com](http://www.BisonAcademy.com)

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# Gamma and Poisson Distribution

Both are extensions of the exponential distribution:

$$p(t) = ae^{-at}$$

Gamma: Time until  $k$  events occur.

- The time until  $k$  customers arrive,
- The time until  $k$  atoms decay,
- The time until you've been invited to  $k$  parties,

Poisson: Probability  $k$  events occur in  $M$  seconds

- The number of pieces of mail you receive each day (the sending time is exponential)
  - The number of cars through in intersection in one minute
  - The number of customers arriving at a restaurant in one hour
  - Also used to approximate binomial distributions where  $n$  is large
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# Gamma Distribution

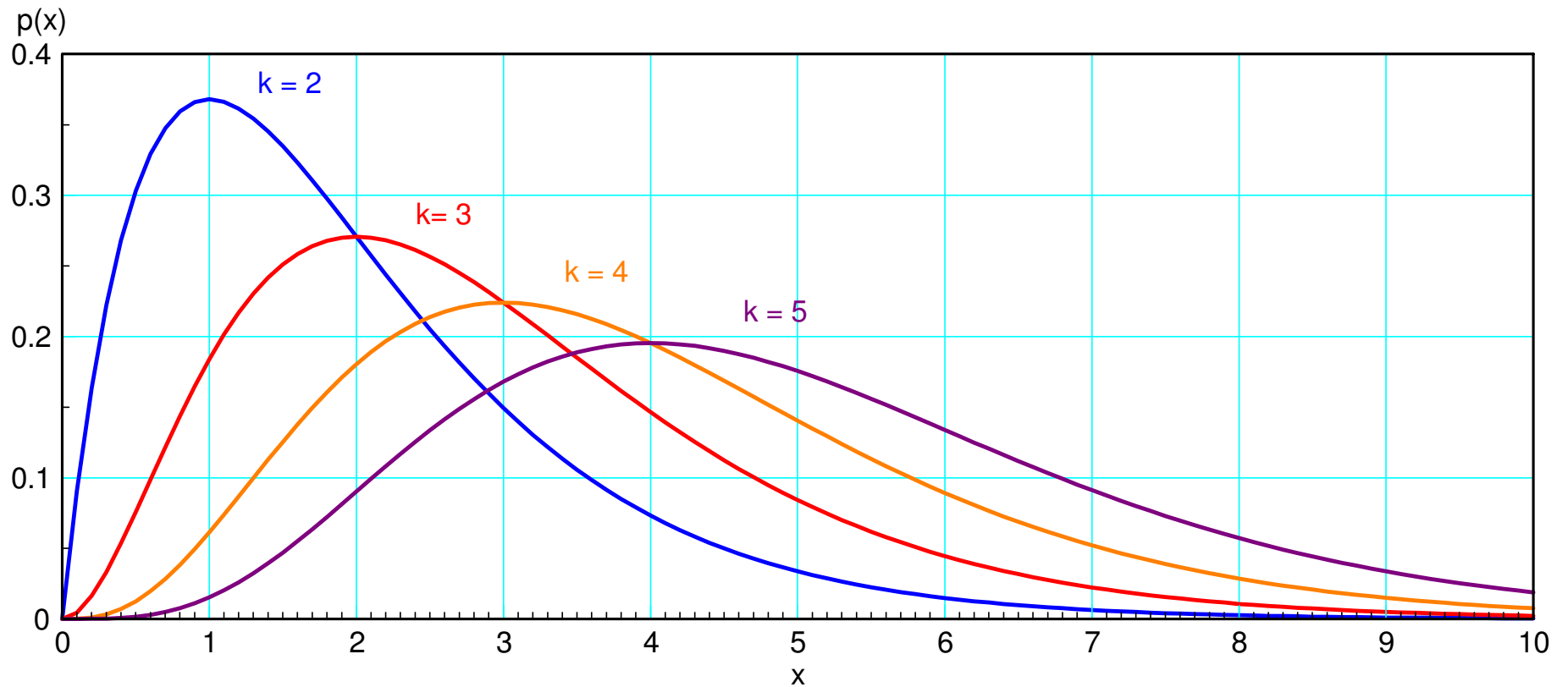
Exponential: Time until next event

Gamma: Time until k events

	Exponential	Gamma
pdf	$f_x = a e^{-ax}$	$f_x = \left( \frac{a^k}{(k-1)!} \right) x^{k-1} e^{-ax}$
cdf	$F_x = 1 - e^{-ax}$	complicated
mgf	$\left( \frac{a}{s+a} \right)$	$\left( \frac{a}{s+a} \right)^k$
mean	$\left( \frac{1}{a} \right)$	$\left( \frac{k}{a} \right)$
variance	$\left( \frac{1}{a^2} \right)$	$\left( \frac{k}{a^2} \right)$

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## pdf of a Gamma distribution:



pdf for a Gamma distribution with an average arrival time of 1

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## Gamma pdf using convolution

Convolution works pretty well in Matlab

- If you can input the pdf for one event in Matlab
- You can compute the pdf for N events using convolution

The heart of a gamma distribution is an exponential distribution

- Probability never goes to zero
- $p(x)$  is *almost* zero at 10x the mean
- $p(x) = 0.000045$

## Convolution Example:

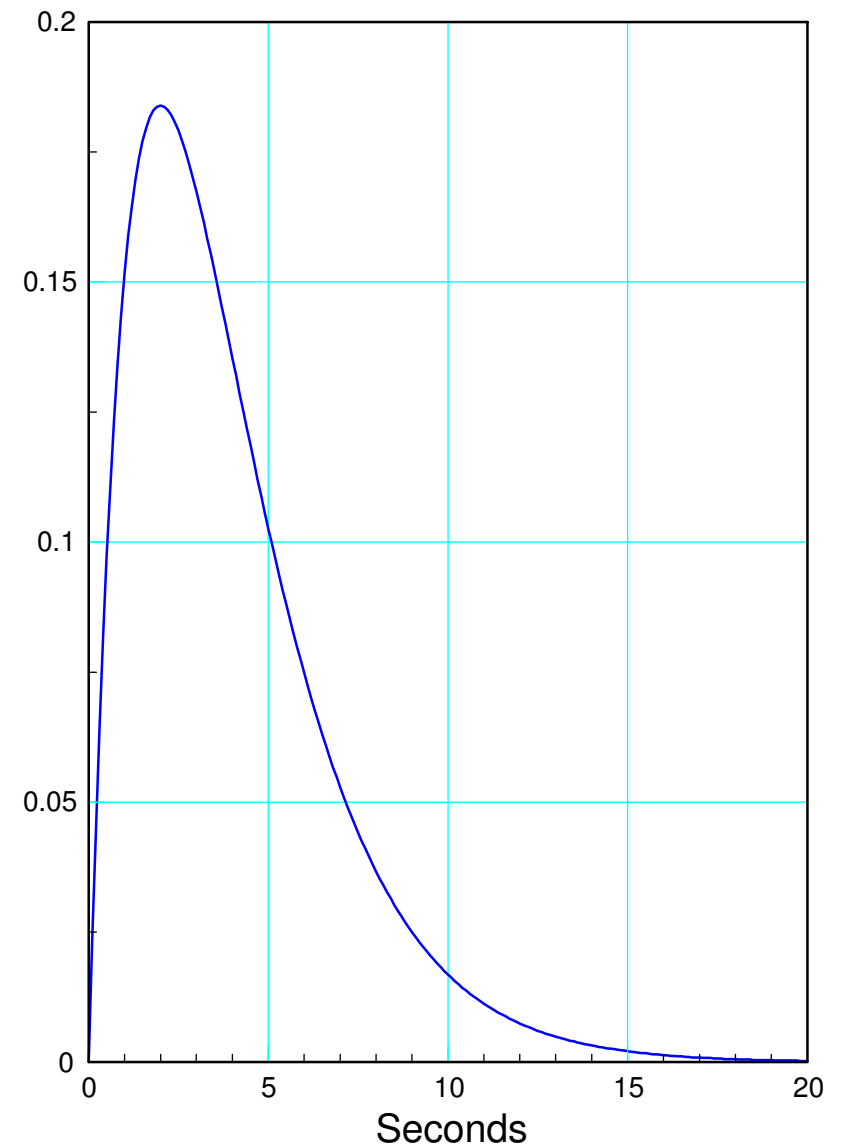
A = The time until 1 event happens

- Exponential distribution
- Mean = 2 minutes

What is the pdf for 2 events?

Matlab Code:

```
dt = 0.01;  
t = [0:dt:20]';  
npt = length(t);  
A = 1/2 * exp(-t/2);  
B = conv(A, A) * dt;  
B = B(1:npt);  
plot(t,B);
```



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## Repeat using LaPlace Transforms

The LaPlace transform of A is

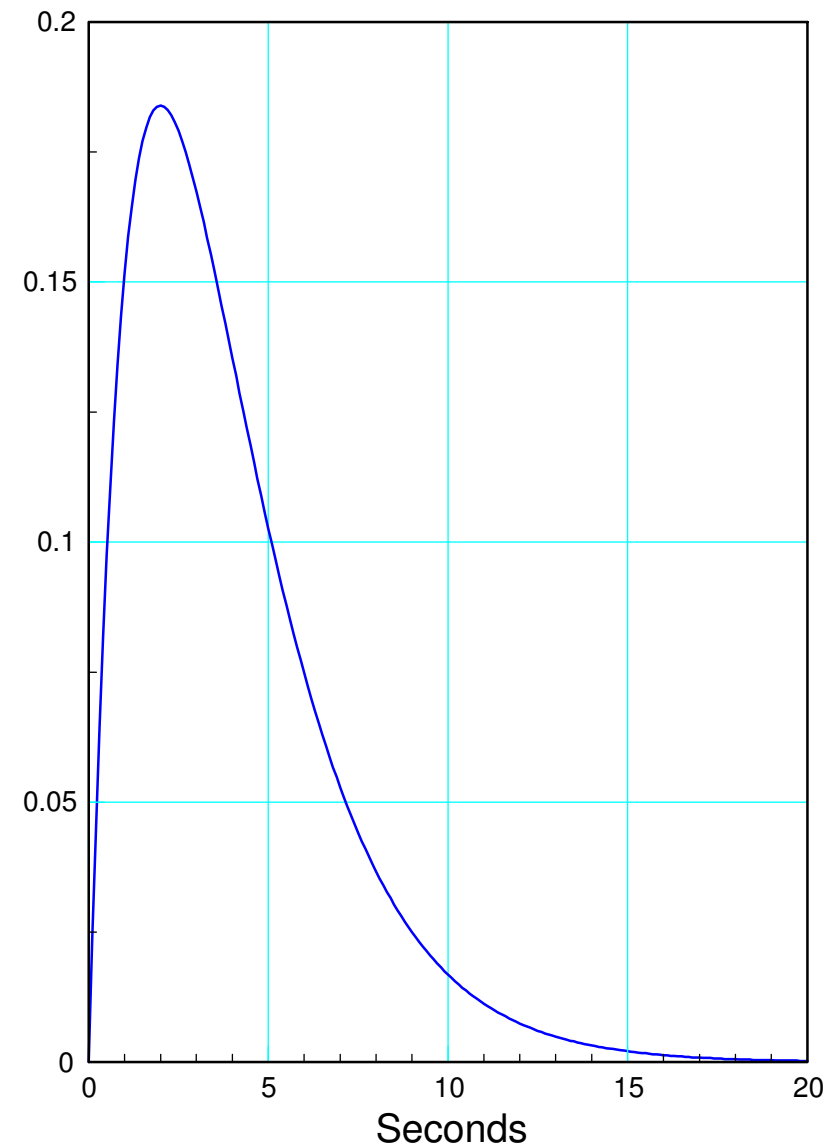
$$A = \left( \frac{1/2}{s+1/2} \right)$$

The LaPlace transform for two events is

$$B = \left( \frac{1/2}{s+1/2} \right)^2$$

Taking the inverse LaPlace transform

$$b(t) = \frac{1}{4} \cdot t \cdot e^{-t/2} \cdot u(t)$$



## Convolution Example #2:

A is the time until 1 event happens

- Exponential distribution
- Mean = 2 minutes

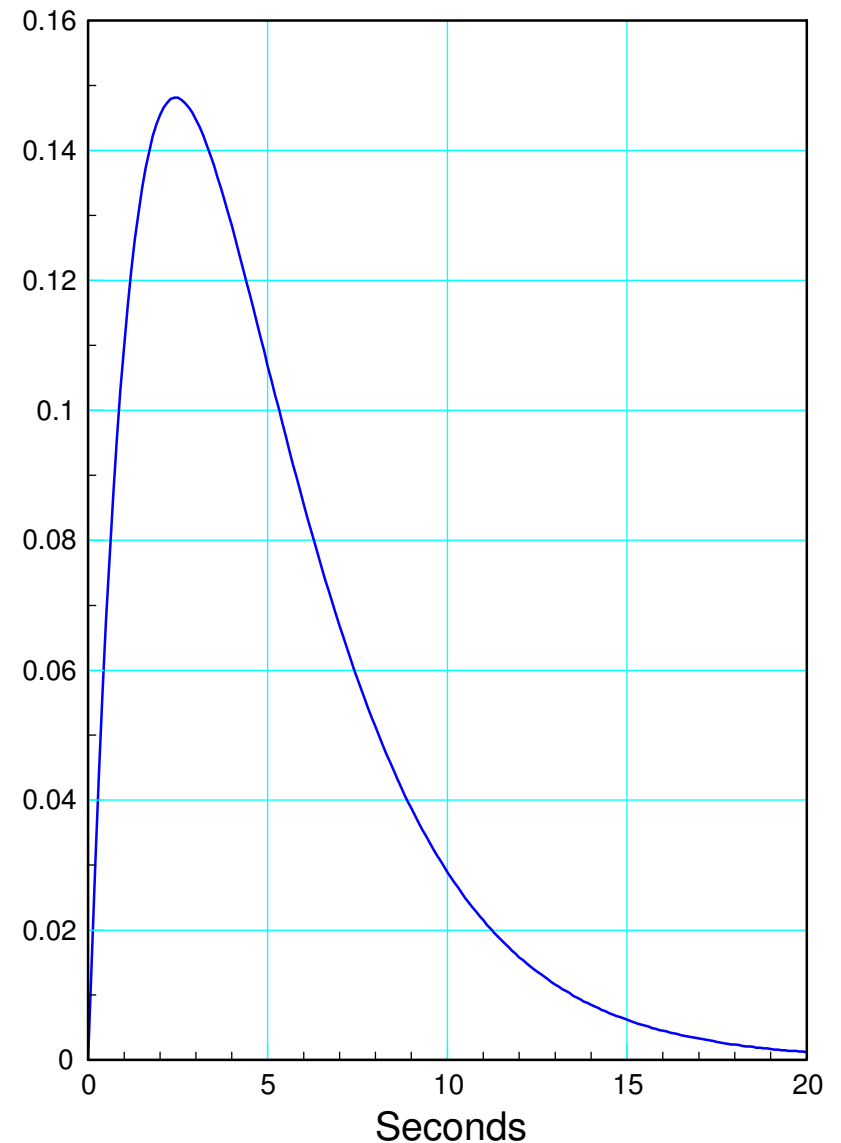
B is the time until 1 event happens

- Exponential distribution
- Mean = 3 minutes

What is the pdf for  $A + B$ ?

Matlab Code:

```
dt = 0.01;  
t = [0:dt:30]';  
npt = length(t);  
A = 1/2 * exp(-t/2);  
B = 1/3 * exp(-t/3);  
C = conv(A, B) * dt;  
C = C(1:npt);  
plot(t,C);
```





## Example #2 LaPlace Transforms

The LaPlace transform for A and B are

$$A = \left( \frac{1/2}{s+1/2} \right)$$

$$B = \left( \frac{1/3}{s+1/3} \right)$$

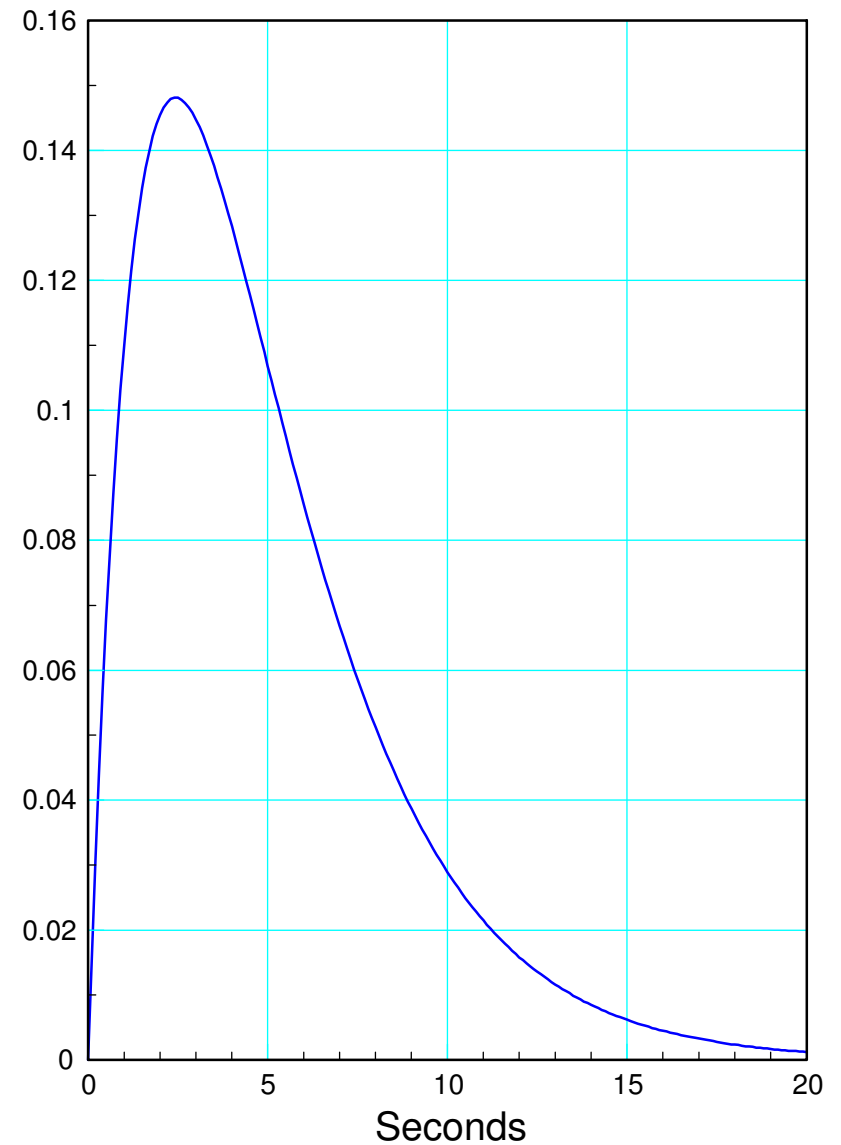
Their sum is

$$C = \left( \frac{1/2}{s+1/2} \right) \left( \frac{1/3}{s+1/3} \right)$$

$$C = \left( \frac{-1}{s+1/2} \right) + \left( \frac{1}{s+1/3} \right)$$

The pdf is thus

$$c(t) = (e^{-t/3} - e^{-t/2})u(t)$$



## Derivation of pdf from mgf:

Example 1: Determine the pdf and cdf for a Gamma distribution with

- $k = 3$ , and
- $a = 0.2$

Solution: The moment generating function is

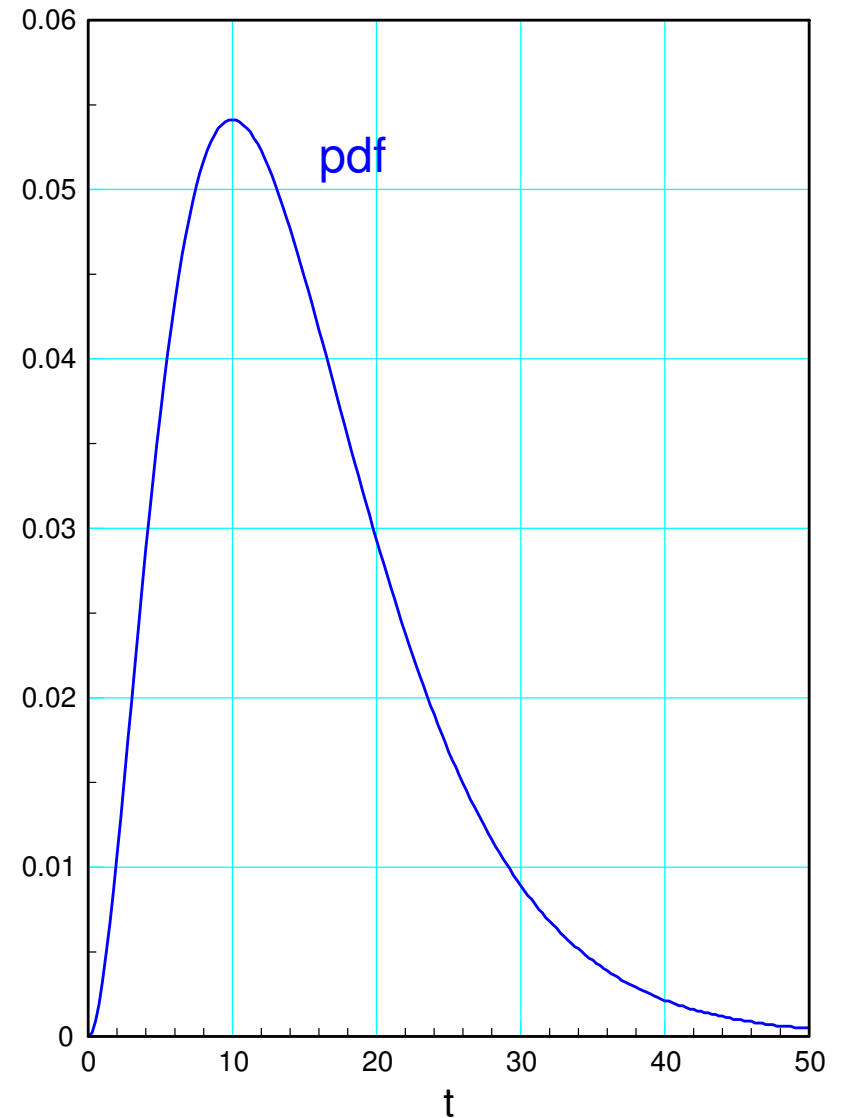
$$\psi(s) = \left( \frac{0.2}{s+0.2} \right)^3$$

From a table of Laplace transforms

$$\left( \frac{1}{s-a} \right)^3 \rightarrow \frac{1}{2!} t^2 e^{at} u(t)$$

Substituting

$$f_x = (0.2)^3 \cdot 2 \cdot \frac{1}{2} t^2 e^{-0.2t} u(t)$$



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The cdf is the integral of the pdf

$$F_X(s) = \left( \frac{0.2}{s+0.2} \right)^3 \left( \frac{1}{s} \right)$$

Doing partial fraction expansion

$$F_X(s) = \left( \frac{a}{s} \right) + \left( \frac{b}{(s+0.2)^3} \right) + \left( \frac{c}{(s+0.2)^2} \right) + \left( \frac{d}{s+0.2} \right)$$

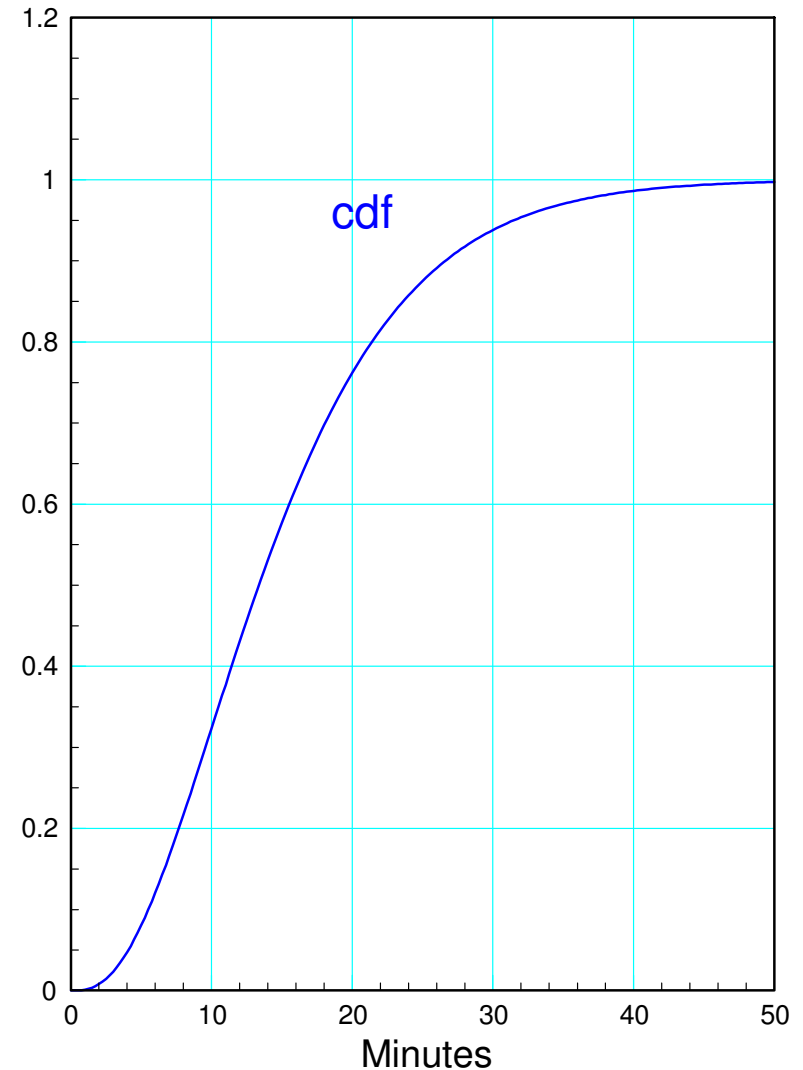
$$F_X(s) = \left( \frac{1}{s} \right) + \left( \frac{-0.04}{(s+0.2)^3} \right) + \left( \frac{-0.2}{(s+0.2)^2} \right) + \left( \frac{-1}{s+0.2} \right)$$

Terms

- {a, b} are from the cover-up method
- {c,d} are from placing over a common denominator

The cdf is thus

$$F_x = (1 + (-0.02 t^2 - 0.2 t - 1) e^{-0.2t}) u(t)$$



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## Gamma Distribution with Learning

Assume you are doing a task

- Each attempt has an exponential distribution

You get better each time (learning is going on)

- 1st attempt: mean = 10 minutes
- 2nd attempt: mean = 9 minutes
- 3rd attempt: mean = 8 minutes

How long will it take you to complete three tasks?

What is the probability you will complete all three tasks within 30 minutes?



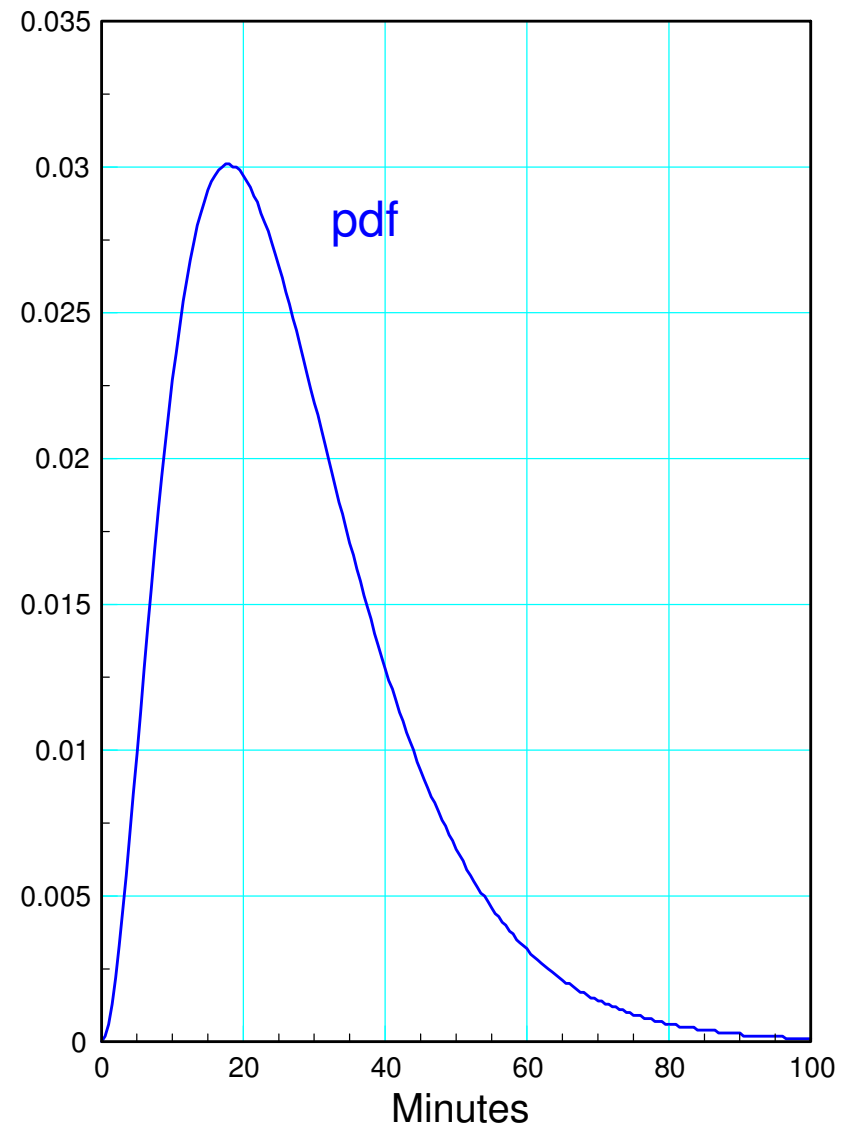
# Solve using Convolution

The longest mean is 10 minutes

- Go out to 100 minutes

## Matlab Code

```
dt = 0.1;  
t = [0:dt:100]';  
A = 1/10 * exp(-t/10);  
B = 1/9 * exp(-t/9);  
C = 1/8 * exp(-t/8);  
AB = conv(A,B) * dt;  
ABC = conv(AB,C) * dt;  
npt = length(t);  
ABC = ABC(1:npt);  
plot(t,ABC);
```



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## Solve Using Moment Generating Functions

This is where moment generating functions really shine:

The LaPlace transform for each task is

$$A(s) = \left( \frac{1/10}{s+1/10} \right) = \left( \frac{1}{10s+1} \right)$$

$$B(s) = \left( \frac{1/9}{s+1/9} \right) = \left( \frac{1}{9s+1} \right)$$

$$C(s) = \left( \frac{1/8}{s+1/8} \right) = \left( \frac{1}{8s+1} \right)$$

The LaPlace transform for completing all three tasks is

$$Y = ABC = \left( \frac{1/10}{s+1/10} \right) \left( \frac{1/9}{s+1/9} \right) \left( \frac{1/8}{s+1/8} \right)$$

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## pdf of 3 tasks

The pdf is the inverse transform.

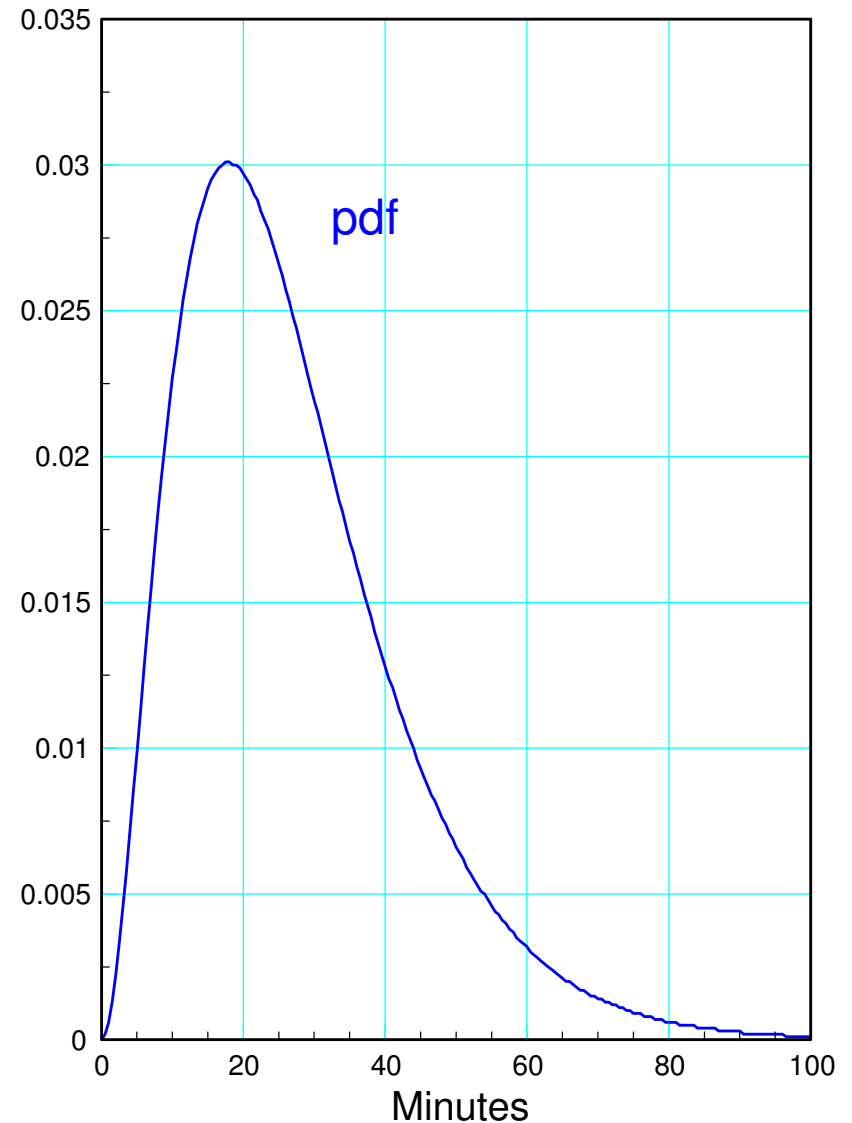
$$Y(s) = \left( \frac{1/10}{s+1/10} \right) \left( \frac{1/9}{s+1/9} \right) \left( \frac{1/8}{s+1/8} \right)$$

Take a partial fraction expansion

$$Y(s) = \left( \frac{5}{s+1/10} \right) + \left( \frac{-9}{s+1/9} \right) + \left( \frac{4}{s+1/8} \right)$$

Take the inverse Laplace transform

$$y(t) = (5e^{-t/10} - 9e^{-t/9} + 4e^{-t/8})u(t)$$



## cdf of Three Tasks

The cdf is the integral of the pdf

In LaPlace

$$cdf = \left(\frac{1}{s}\right) pdf$$

or

$$cdf = \left(\frac{1}{s}\right) \left(\frac{1/10}{s+1/10}\right) \left(\frac{1/9}{s+1/9}\right) \left(\frac{1/8}{s+1/8}\right)$$

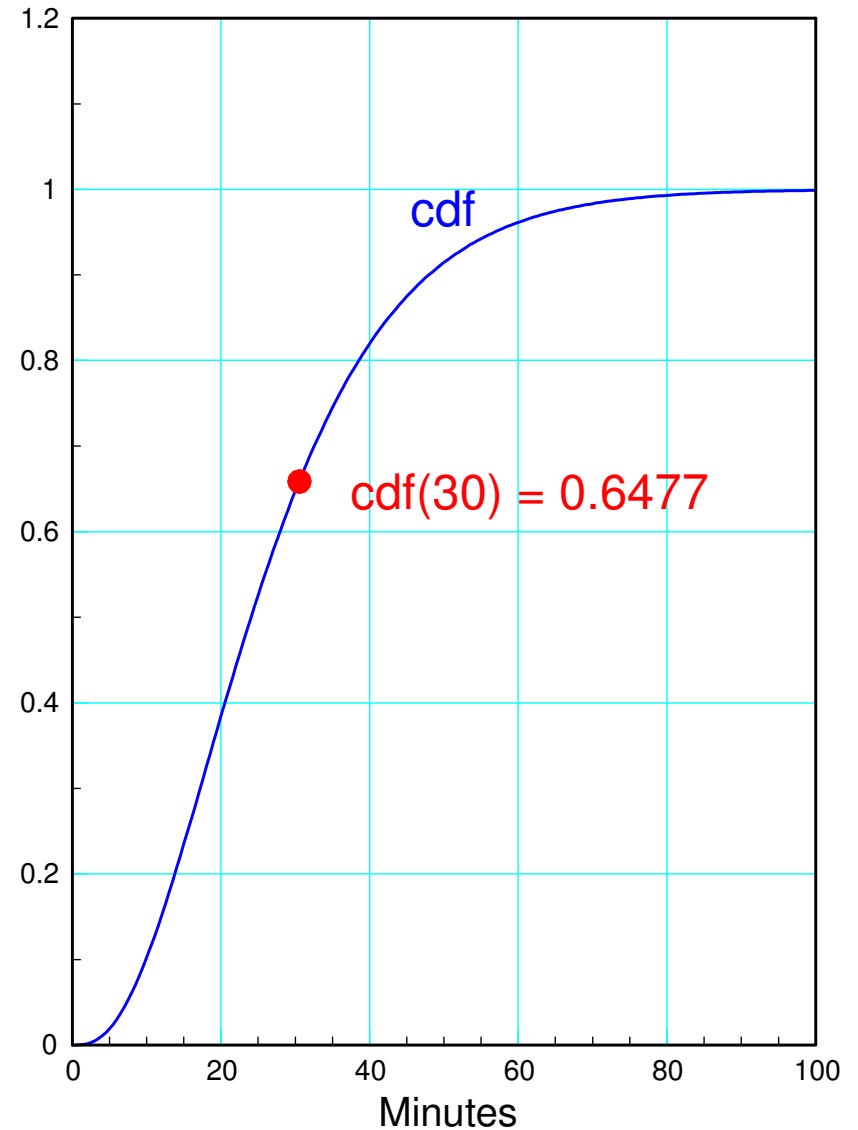
Do a partial fraction expansion

$$cdf = \left(\frac{1}{s}\right) + \left(\frac{-50}{s+1/10}\right) + \left(\frac{81}{s+1/9}\right) + \left(\frac{-32}{s+1/8}\right)$$

gives

$$cdf = (1 - 50e^{-t/10} + 81e^{-t/9} - 32e^{-t/8})u(t)$$

$$cdf(30) = 0.6477$$





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## Poisson Distribution.

The Poisson distribution is actually a discrete probability function.

- Gamma: The time until the  $k$ th customer arrives, ( Gamma )
- Poisson: The probability that  $k$  customers will arrive in a fixed interval

The Poisson distribution is useful if you want to know

- How many cars will go through an intersection in one hour,
  - How many customers will arrive in one hour,
  - How many patients will go to the emergency room in one day, or
  - The number of times your boss will notice you over the course of one week.
  - The probability of a binomial distribution when  $n$  is large
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# Assumptions

(Wikipedia)

- $k$  is the number of times an event occurs in an interval and  $k$  can take values 0, 1, 2, ....
  - The occurrence of one event does not affect the probability that a second event will occur. That is, events occur independently.
  - The average rate at which events occur is independent of any occurrences. For simplicity, this is usually assumed to be constant, but may in practice vary with time.
  - Two events cannot occur at exactly the same instant; instead, at each very small subinterval exactly one event either occurs or does not occur.
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The pdf for a Poisson distribution is (wikipedia)

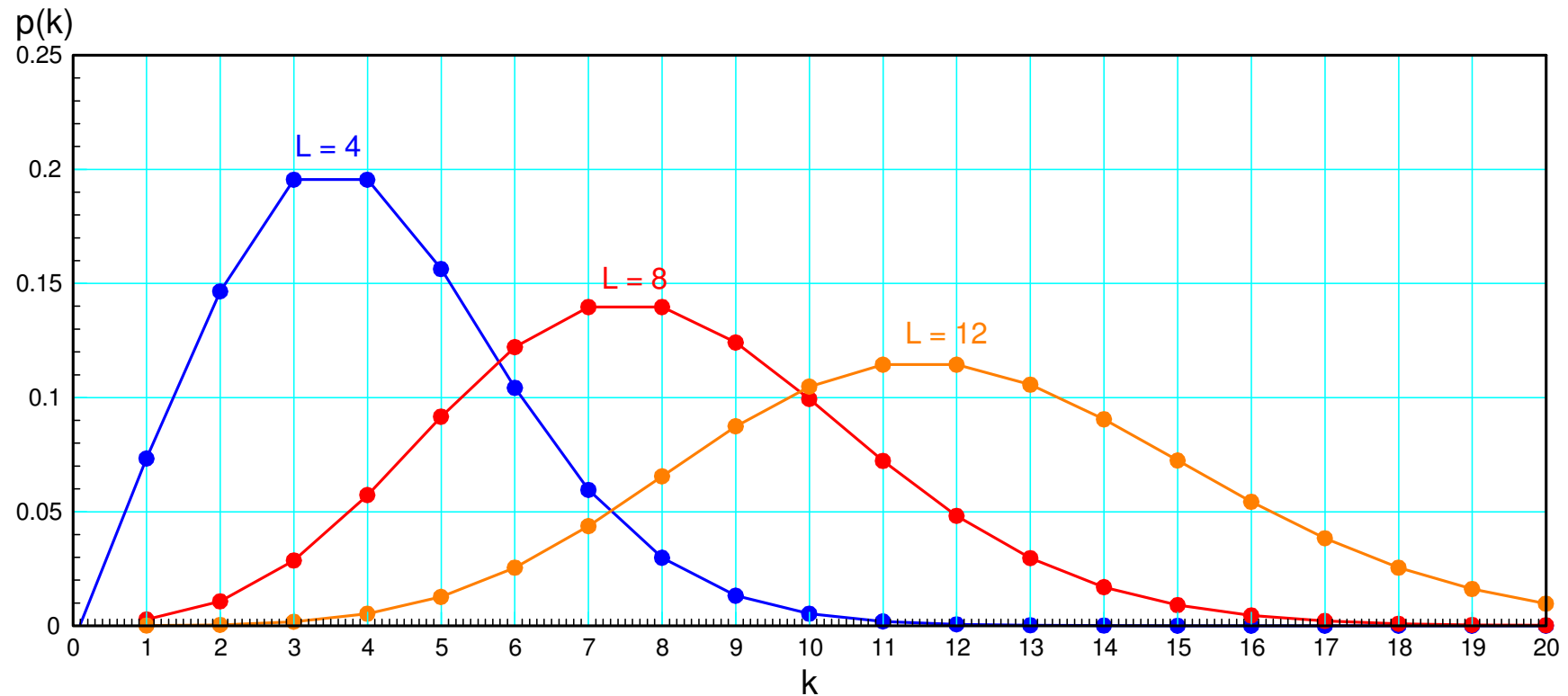
	Exponential	Poisson
pdf	$f_x = a e^{-ax}$	$f(k, \lambda) = \frac{\lambda^k e^{-\lambda}}{k!}$
cdf	$F_x = 1 - e^{-ax}$	complicated
mgf	$\left( \frac{a}{s+a} \right)$	$\exp(\lambda(z-1))$
mean	$\left( \frac{1}{a} \right)$	$\lambda$
variance	$\left( \frac{1}{a^2} \right)$	$\lambda$

Here,  $\lambda = 1/a$  for equating exponential and Poisson processes.

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The pdf for a Poisson distribution looks like the following ( $\lambda = 10$  ( $a = 1/10$ ) for illustration purposes).



pdf for a Poisson distribution with  $\lambda = \{4, 8, 12\}$ .  
Note that this is a discrete pdf - so only the integer values of  $k$  matter

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# Poisson Approximation for a Binomial Distribution

A Poisson distribution is also a good approximation for a binomial distribution where

- $n$  is large
- $\lambda = np$

**Example 1:** The pdf for a binomial distribution

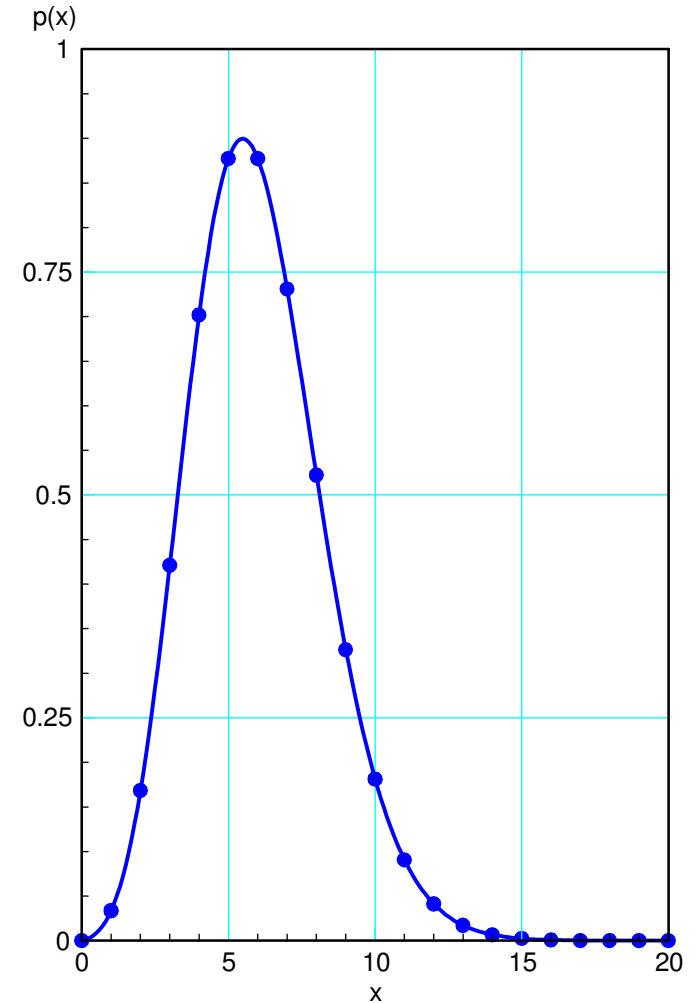
$$n = 100, \quad p = 0.05, \quad \lambda = np = 5$$

Binomial:

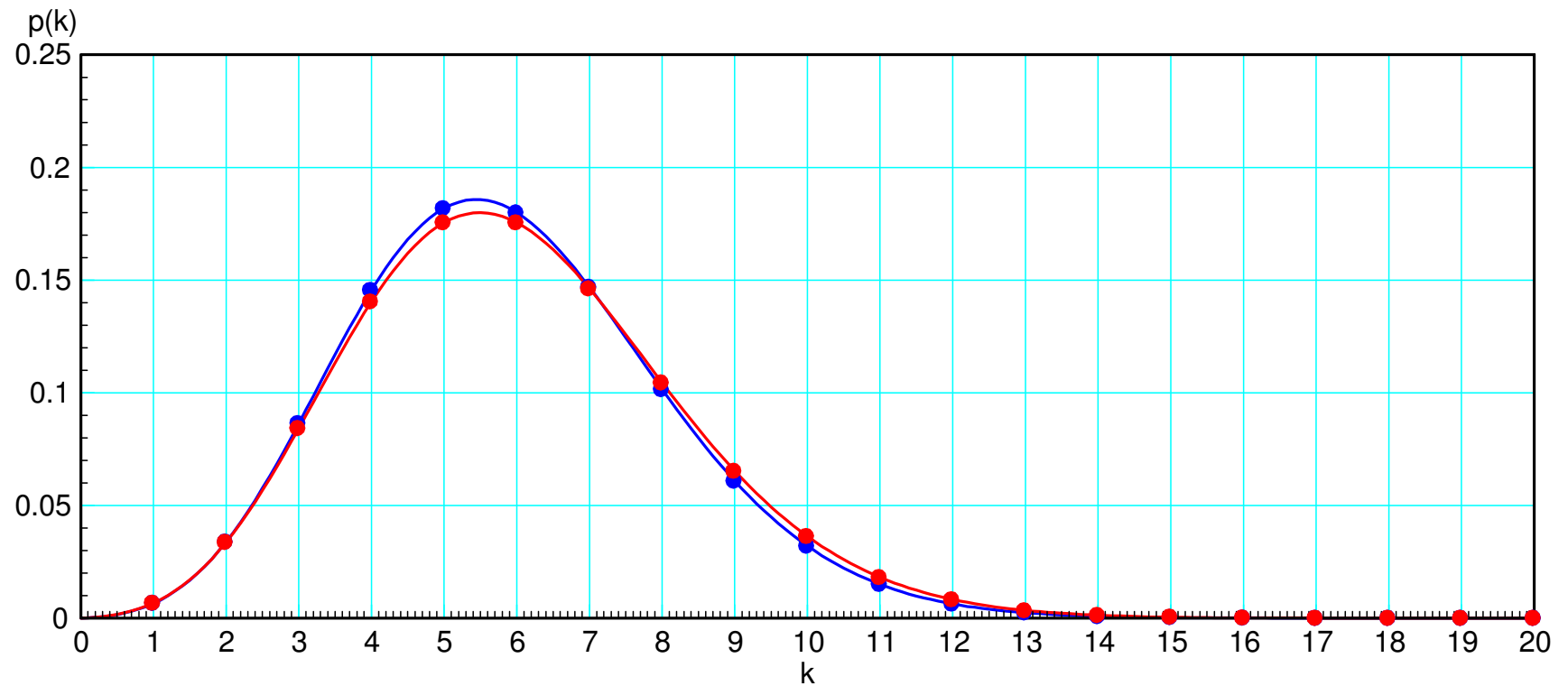
$$f_1(x) = \binom{100}{x} (0.05)^x (0.95)^{100-x}$$

Poisson (  $\lambda = np = 5$  )

$$f_2(x) = \frac{1}{x!} \cdot 5^x \cdot e^{-5}$$



# Poisson vs. Binomial



Binomial (blue) vs. Poisson (red) with  $np = 5$

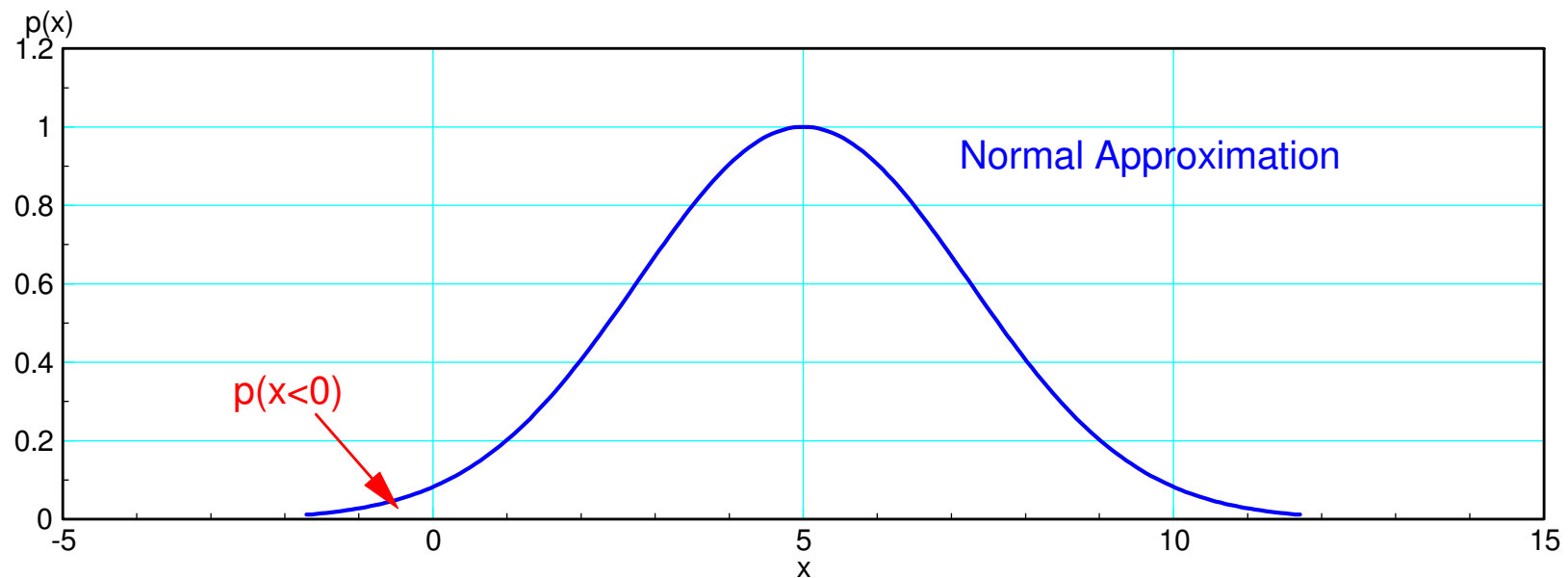
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A Poisson approximation is a slightly more complicated approximation for a binomial distribution than a Normal approximation. It's more accurate however.

- A normal distribution goes from  $-\infty$  to  $+\infty$
- A Poisson distribution is zero for  $k < 0$

In the case of a binomial distribution, you'll never get a negative total.

- A Poisson approximation is slightly more accurate.



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**Example 2:** Plot the probability density function for a binomial distribution with

$$n = 10,000 \quad p = 0.0005 \quad np = 5$$

A Binomial pdf has numeric problems:

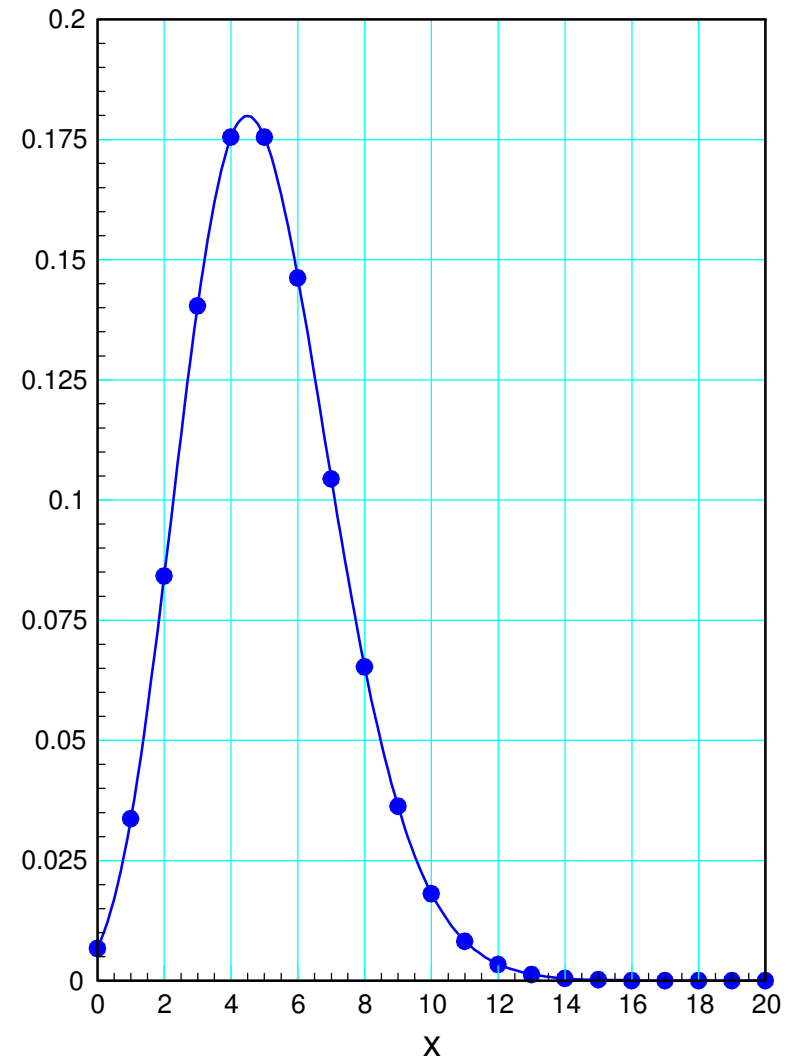
- $10,000!$  is out of range

$$f(x) = \binom{10,000}{x} (0.0005)^x (0.9995)^{10,000-x}$$

Not a problem with a Poisson approximation

- $np = 5$

$$f(x) \approx \frac{1}{x!} \cdot 5^x \cdot e^{-5}$$





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## Summary

A Gamma distribution is an exponential distribution

- Where you wait until  $N$  events occur
- The moment generating function is  $\left(\frac{a}{s+a}\right)^N$

A Poisson distribution

- Is an exponential distribution where you count how many events occur over a time interval
- Is a good approximation for a binomial distribution