Exponential Distribution

ECE 341: Random Processes

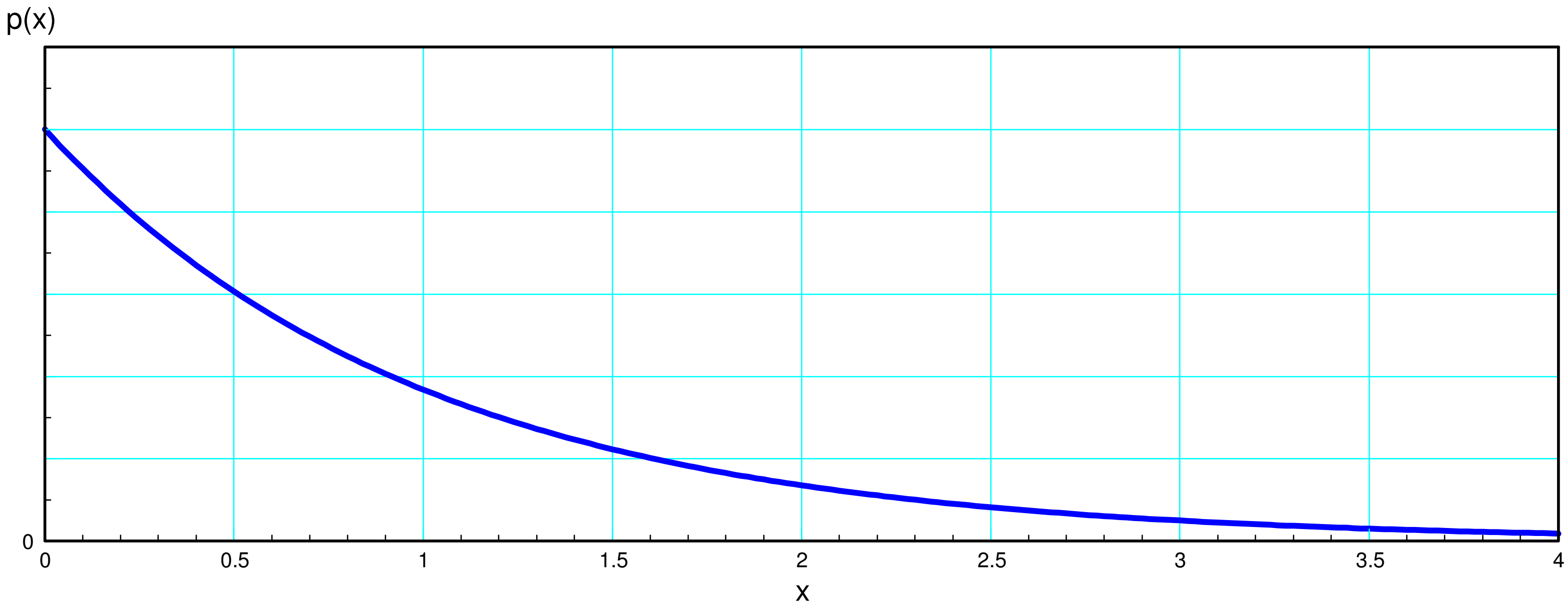
Lecture #15

note: All lecture notes, homework sets, and solutions are posted on www.BisonAcademy.com

Exponential Distribution

Another common continuous probabilty distribution is the exponential distribution. With this distribution, the pdf is exponential:

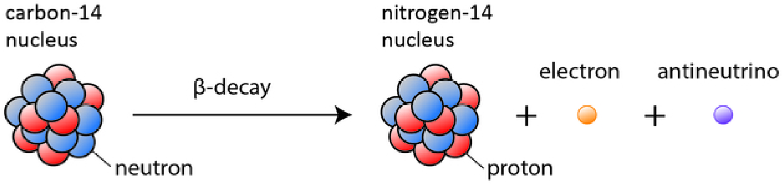
$$f_X(x) = \begin{cases} a\, e^{-ax} & 0 < x < \infty \\ 0 & otherwise \end{cases}$$



pdf for an exponential distribution

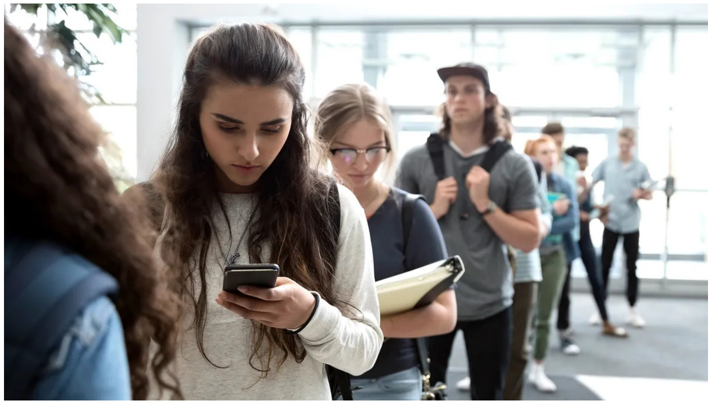
Examples of where this type of distibution is encountered is:

- Probability that the length of a telephone call is less than x minutes
- Probability that the next atom will decay within x seconds
- Time it takes for the next customer to arrive at a store
- Time it takes to serve the next customer



Exponential distributions also lead into queueing theory:

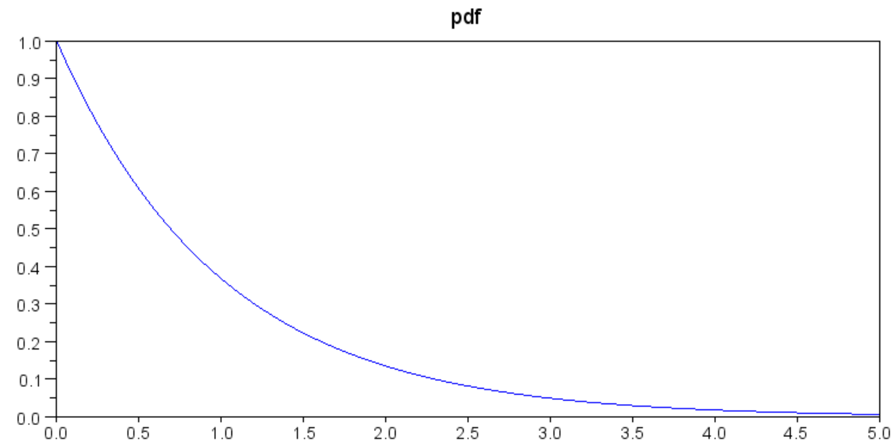
- How long a customer will have to wait in line to be served.



Parameters of Exponential Distributions

pdf: As stated before

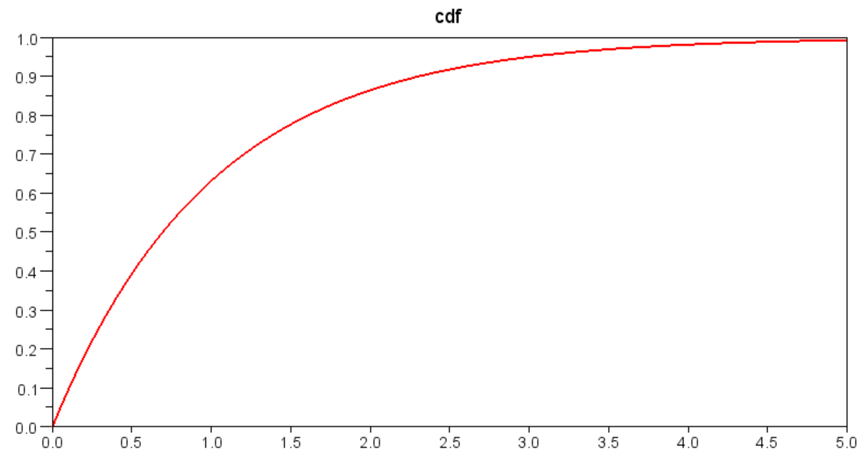
$$pdf(x) = a\, e^{-ax}$$



cdf: The cdf is the integral of the pdf

$$cdf(x) = \int_0^x (a\, e^{-at})\, dt$$

$$cdf(x) = 1 - e^{-ax} \qquad x > 0$$



Moment Generating Function:

$$\psi(s) = \left(\frac{a}{s+a}\right)$$

To be a valid probability distribution:

$$m_0 = \psi(s=0) = 1 = \left(\frac{a}{a}\right)$$

Mean: The mean of an exponential distribution is

$$\mu = \int_0^\infty p(x)\, x\, dx = \int_0^\infty (ae^{-ax})\, x\, dx = \left(\frac{1}{a}e^{-ax}(-ax-1)\right)\Bigg|_0^\infty = \left(\frac{1}{a}\right)$$

$$\mu = m_1 = -\psi'(0) = \left(\frac{a}{(s+a)^2}\right)\Bigg|_{s=0} = \frac{1}{a}$$

Variance: Use the moment generating function - it's a lot easier

$$m_2 = -\psi''(0) = -\frac{d}{ds}\left(\frac{a}{(s+a)^2}\right)$$

$$m_2 = -\frac{d}{ds}\left(\frac{a}{(s+a)^2}\right)_{s=0} = \left(\frac{2a}{(s+a)^3}\right)_{s=0} = \left(\frac{2}{a^2}\right)$$

so

$$\sigma^2 = m_2 - m_1^2$$

$$\sigma^2 = \left(\frac{2}{a^2}\right) - \left(\frac{1}{a}\right)^2 = \left(\frac{1}{a^2}\right)$$

Matlab Example:

Use the cdf:

$$cdf(x) = \int_0^x pdf(t)\,dt = (1 - e^{-ax})u(x)$$

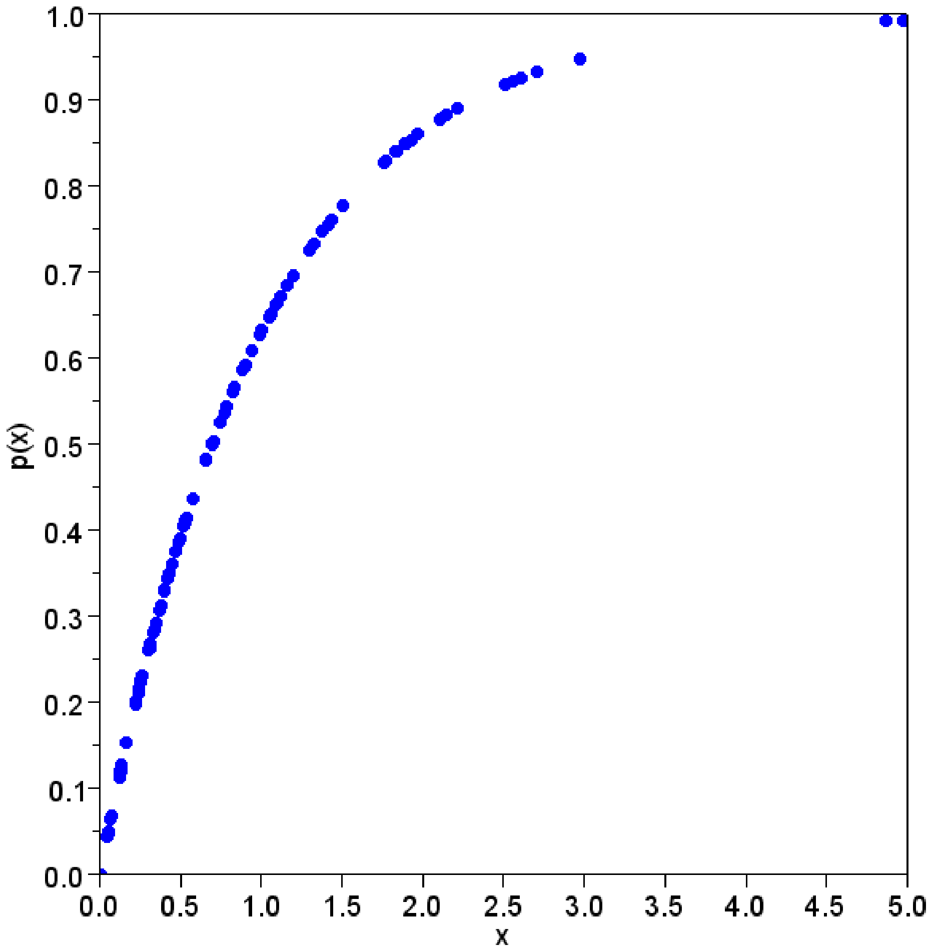
$$x = -\left(\frac{1}{a}\right)\ln(1-p)$$

In matlab

```
p = rand(10,1);

a = 1;
x = -(1/a) * log(1-p)
```

p	x
0.6862	1.1590
0.9593	3.2004
0.4850	0.6635
0.7880	1.5511
0.7550	1.4066
0.7228	1.2831
0.2646	0.3074
0.6884	1.1660
0.9323	2.6930



Example: A block of radioactive material is sitting next to a Geiger counter. In 30 minutes, the Geiger counter detects 100 atoms decaying. Determine the pdf and CDF for the time until the next atom decays (in minutes).

Solution: The average number of atoms decaying per minute is

$$\bar{x} = \frac{100 \text{ atoms}}{30 \text{ minutes}} = 3.33 \frac{\text{atoms}}{\text{min}} = \frac{1}{a}$$

$$a = 0.3$$

pdf:

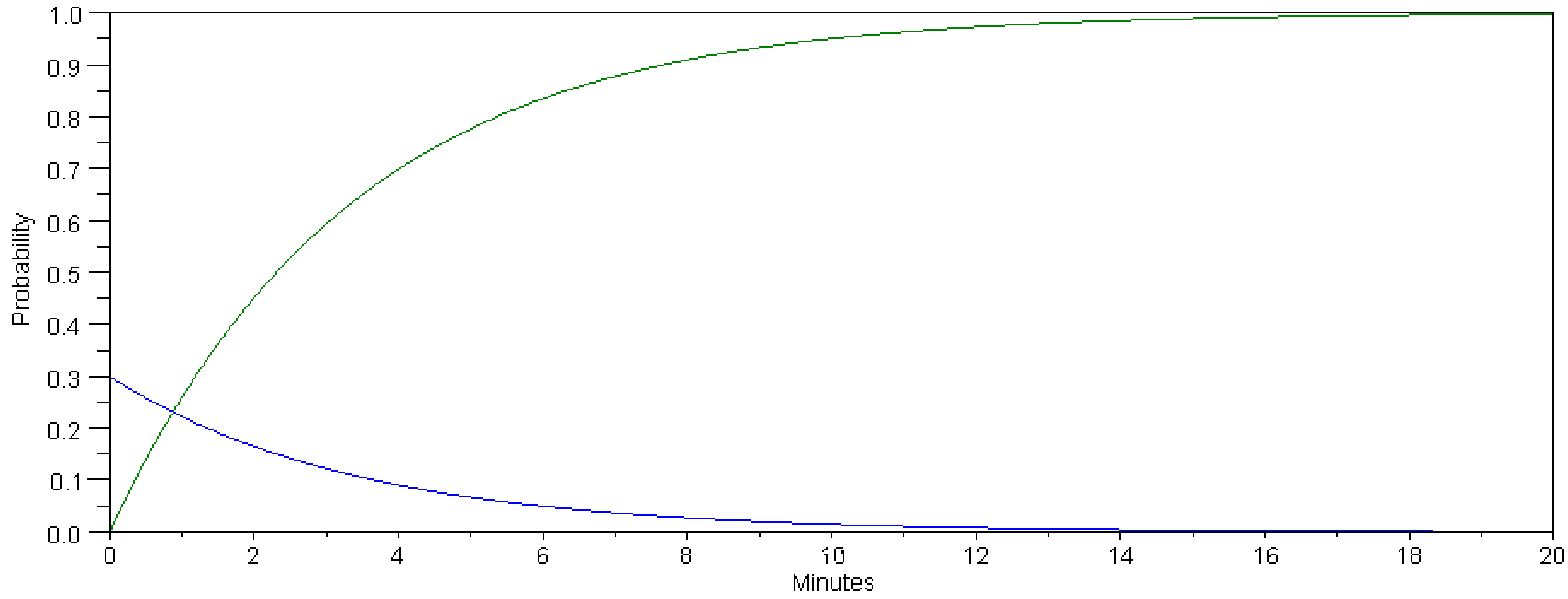
$$f_X(x) = \begin{cases} 0.3 \cdot e^{-0.3x} & 0 < x < \infty \\ 0 & otherwise \end{cases}$$

CDF:

$$F_X(x) = \begin{cases} 1 - e^{-0.3x} & 0 < x < \infty \\ 0 & otherwise \end{cases}$$

Matlab

```
x = [0:0.01:20]';
p = 0.3 * exp(-0.3*x);
C = 1 - exp(-0.3*x);
plot(x,p,x,C)
xlabel('Minutes');
ylabel('Probability');
```



pdf (blue) and CDF (green) for time until you detect an atom decaying

Queueing Theory

- Application of exponential distributions
- How many servers are needed at a restaraunt?

For example, assume

- Customers arrive at a fast-food restaraunt with
  - An exponential distribution, with
  - An average of one customer every minute.
- The time it takes to serve a customer is
  - An exponential distribution with
  - An average of 30 seconds per customer

How long will the longest wait be for a 1 hour shift?

Start with the first customer

Matlab Script Window

```
N = 5;

Tarr = zeros(N,1);
Tser = zeros(N,1);
Tdone = zeros(N,1);
Twait = zeros(N,1);
Queue = zeros(N,1);

% Start with the first customer
n = 1;
Tarr(n) = -60*log(1 - rand);
Tser(n) = Tarr(n);
Twait(n) = 0;
Tdone(n) = Tser(n) -30*log(1-rand);
Queue(n) = 0;

[Tarr, Tser, Twait, Tdone, Queue]
```

Matlab Command Window

Tarr	Tser	Twait	Tdone	Q
68.77	68.77	0	196.52	0
0	0	0	0	0
0	0	0	0	0
0	0	0	0	0
0	0	0	0	0

Add more customers

Tarr: time of previous customer + exponential time with mean of 60 seconds

Tser: The time customer is served. Whichever is larger

- Time customer arrived
- Time previous customer was done

Tdone: Time customer was served + Time it takes to finish that customer's order

Matlab Script Window	Matlab Command Window

```
for n=2:N
   Tarr(n) = Tarr(n-1) - 60*log(1-rand);
   Tser(n) = max(Tarr(n), Tdone(n-1));
   Tdone(n) = Tser(n) - 30*log(1-rand);
end

[Tarr, Tser, Twait, Tdone, Queue]
```

arrive	serve	wait	done	queue
31.0	31.0	0	50.3	0
45.1	50.3	0	83.2	0
130.1	130.1	0	145.4	0
140.4	145.4	0	**166.2**	0
146.1	**166.2**	0	201.9	0

Compute Wait Time & Queue Size

Wait time

- Time from arriving to being served

Queue Size

- Number of customers still there when you arrive
- Their finish time is after you arrived

Matlab Script Window

```
for n=2:N
  Tarr(n) = Tarr(n-1) - 60*log(1-rand);
  Tser(n) = max(Tarr(n), Tdone(n-1));
  Tdone(n) = Tser(n) - 30*log(1-rand);
  Twait(n) = Tser(n) - Tarr(n);
  Queue(n) = sum( Tdone(1:n-1)>Tarr(n));
end
```

Matlab Command Window

arrive	serve	wait	done	queue
10.8	10.8	0	12.8	0
91.6	91.6	0	**160.7**	0
107.6	**160.7**	**53.0**	**168.2**	1
141.4	168.2	26.7	169.7	**2**
208.1	208.1	0	212.0	0

Total Matlab Code

Matlab Script Window

```
N = 30;

Tarr = zeros(N,1);
Tser = zeros(N,1);
Tdone = zeros(N,1);
Twait = zeros(N,1);
Queue = zeros(N,1);

% Start with the first customer
n = 1;
Tarr(n) = -60*log(1-rand);
Tser(n) = Tarr(n);
Twait(n) = 0;
Tdone(n) = Tser(n) -30*log(1-rand);
Queue(n) = 0;
for n=2:N
  Tarr(n) = Tarr(n-1) - 60*log(1-rand);
  Tser(n) = max(Tarr(n), Tdone(n-1));
  Tdone(n) = Tser(n) - 30*log(1-rand);
  Twait(n) = Tser(n) - Tarr(n);
  Queue(n) = sum( Tdone(1:n-1)>Tarr(n));
end

[Tarr, Tser, Twait, Tdone, Queue]
```

Matlab Command Window

arrive	serve	wait	done	queue
0.3	0.3	0	8.2	0
10.1	10.1	0	26.1	0
33.4	33.4	0	134.4	0
116.6	134.4	17.8	160.4	1
209.0	209.0	0	220.1	0
229.2	229.2	0	233.7	0
239.5	239.5	0	253.8	0
245.7	253.8	8.1	265.0	1
343.4	343.4	0	387.5	0
358.9	387.5	28.6	417.5	1
429.5	429.5	0	438.0	0
479.8	479.8	0	512.1	0
540.4	540.4	0	543.5	0
590.8	590.8	0	674.2	0
610.8	674.2	63.4	755.4	1
655.7	755.4	**99.8**	759.3	2
671.0	759.3	88.3	763.4	**3**
835.4	835.4	0	871.1	0
854.7	871.1	16.4	892.7	1
857.9	892.7	34.8	956.8	2

Data Analysis

The maximum wait time is important

- If customers wait too long, they'll leave
  - *lost revenue*
- If the max queue size is too large, customers will turn around and leave
  - *lost revenue*

If either of these numbers get too large

- Hire more servers
- Open more cash registers

If these numbers are too small

- Tell servers to check out
- Reassign people working the cash registers

Matlab Command Window

```
>> max(Twait)

ans =  187.7314

>> max(Queue)

ans =     4
```

Waiting Time

When customers arrive faster than it takes to complete an order, the waiting time increases without limit

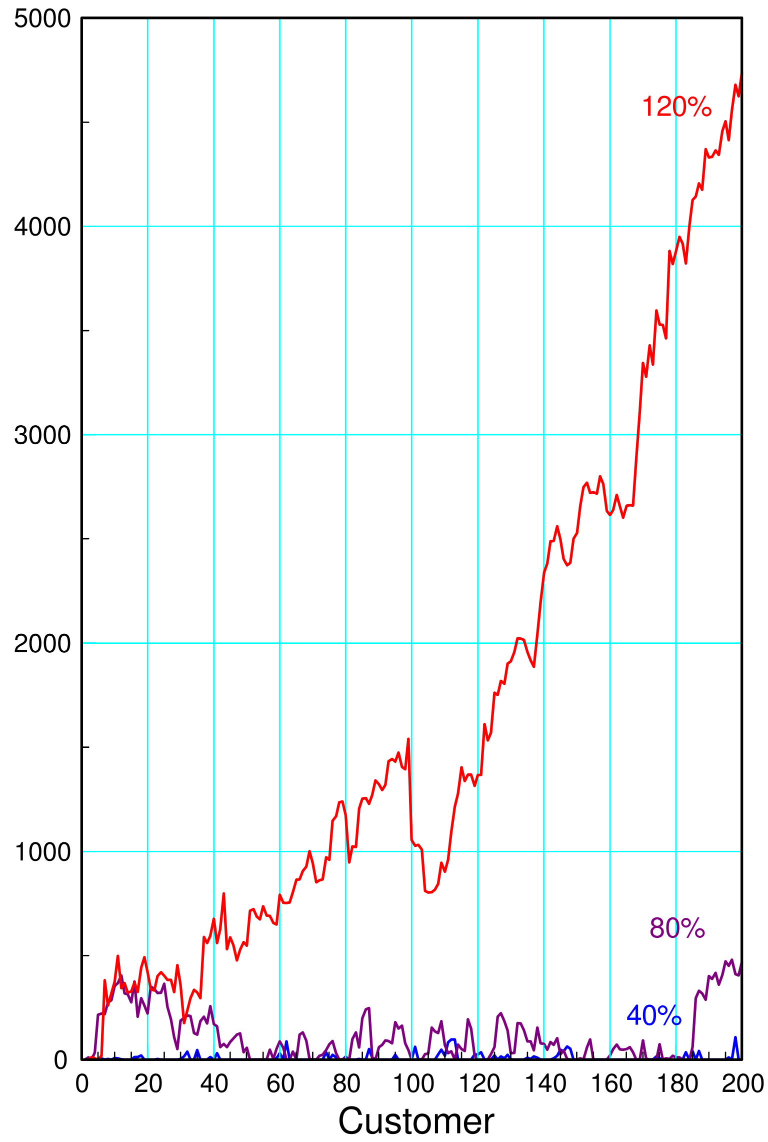
- red line

When orders are filled 20% faster than customers arrive,

- Max wait = 478 seconds
- purple line

When orders are filled 60% faster

- Max wait = 108 seconds
- blue line



Max Queue Size

When orders are filled 20% slower than customers arrive

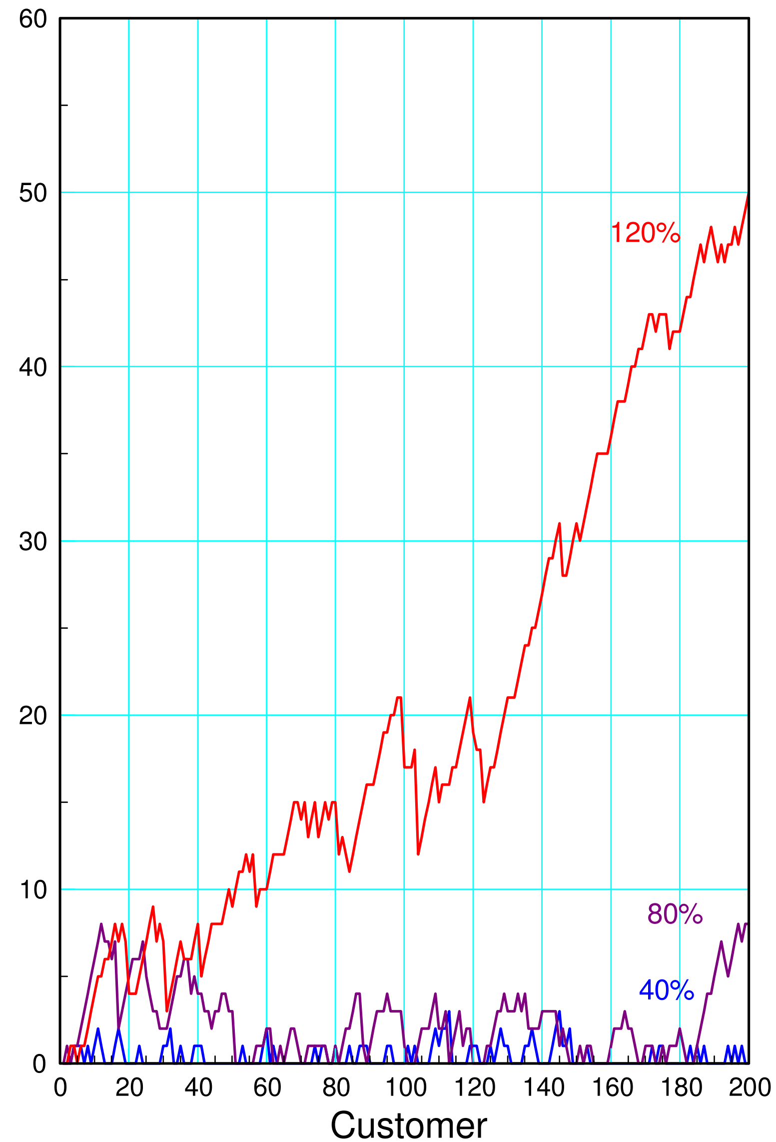
- The queue size grows
- red line

When orders are filled 20% faster

- The max queue size was 8
- purple line

When orders are filled 60% faster

- The max queue size was 3
- blue line



Summary

Exponential distributions model many systems

- Time of a phone call
- Time until an atom decays
- Time until the next customer arrives

Queeing theory is based upon exponential distributions

- Used to determine expected wait times queue sizes in stores and restaraunts