
LaPlace Transforms

ECE 341: Random Processes

Lecture #12

note: All lecture notes, homework sets, and solutions are posted on www.BisonAcademy.com

Transfer Functions and Differential Equations:

LaPlace transforms assume all functions are in the form of

$$y(t) = \begin{cases} a \cdot e^{st} & t > 0 \\ 0 & \text{otherwise} \end{cases}$$

This results in the derivative of y being:

$$\frac{dy}{dt} = s \cdot y(t)$$

This lets you convert differential equations into transfer functions and back.

Example 1: Find the transfer function from X to Y

$$\frac{d^3y}{dt^3} + 6\frac{d^2y}{dt^2} + 11\frac{dy}{dt} + 6y = 8\frac{dx}{dt} + 10x$$

Solution: Substitute 's' for $\frac{d}{dt}$

$$s^3Y + 6s^2Y + 11sY + 6Y = 8sX + 10X$$

Solve for Y

$$(s^3 + 6s^2 + 11s + 6)Y = (8s + 10)X$$

$$Y = \left(\frac{8s+10}{s^3+6s^2+11s+6} \right) X$$

The transfer function from X to Y is

$$G(s) = \left(\frac{8s+10}{s^3+6s^2+11s+6} \right)$$

- Note: The transfer function is often called 'G(s)' since it is the gain from X to Y.
-

Example 2: Find the differential equation relating X and Y

$$Y = \left(\frac{8s+10}{s^3+6s^2+11s+6} \right) X$$

Cross multiply:

$$(s^3 + 6s^2 + 11s + 6)Y = (8s + 10)X$$

Note that 'sY' means 'the derivative of Y'

$$\frac{d^3y}{dt^3} + 6\frac{d^2y}{dt^2} + 11\frac{dy}{dt} + 6y = 8\frac{dx}{dt} + 10x$$

Sidelight: Fractional powers are not allowed in transfer functions.

- s^2Y means 'the second derivative of Y'.
- $s^{2.3}Y$ means 'the 2.3th derivative of Y'.

I have no idea what a 0.3 derivative is.

Solving Transfer Functions with Sinusoidal Inputs

Example 3: Find $y(t)$ given

$$Y = \left(\frac{8s+10}{s^3+6s^2+11s+6} \right) X$$

and

$$x(t) = 3 \cos(4t)$$

Solution: From Euler's identity:

$$e^{j4t} = \cos(4t) + j \sin(4t)$$

$$(a + jb) e^{j4t} = (a \cos(4t) - b \sin(4t)) + j(\cdot)$$

$$a + jb \leftrightarrow a \cos(4t) - b \sin(4t)$$

- real = cosine
 - -imag = sine
-

Take the real part and you the sine wave.

$$s = j4$$

$$X = 3 + j0$$

Convert to phasors

$$Y = \left(\frac{8s+10}{s^3+6s^2+11s+6} \right) X$$

$$Y = \left(\frac{8s+10}{s^3+6s^2+11s+6} \right)_{s=j4} (3 + j0)$$

$$Y = (-0.191 - j0.315)(3 + j0)$$

$$Y = -0.544 - j0.946$$

meaning

$$y(t) = -0.544 \cos(4t) + 0.946 \sin(4t)$$

Example 4: Find $y(t)$ if

$$x(t) = 5 \cos(20t) + 6 \sin(20t)$$

Solution:

$$Y = \left(\frac{8s+10}{s^3+6s^2+11s+6} \right)_{s=j20} (5 - j6)$$

$$Y = (-0.019 - j0.005)(5 - j6)$$

$$Y = -0.123 + j0.092$$

meaning

$$y(t) = -0.123 \cos(20t) - 0.092 \sin(20t)$$

Note that the gain varies with frequency (i.e. this is a filter).

Example 5: Find $y(t)$ if

$$x(t) = 3 \cos(4t) + 5 \cos(20t) + 6 \sin(20t)$$

Solution: Treat this as two separate problems

- $x(t) = 3 \cos(4t)$
- $x(t) = 5 \cos(20t) + 6 \sin(20t)$

The total input is the sum of the two $x(t)$'s.

The total output is the sum of the two $y(t)$'s

$$y(t) = -0.544 \cos(4t) + 0.946 \sin(4t) \\ -0.123 \cos(20t) - 0.092 \sin(20t)$$

Solving Transfer Functions with Step Inputs

Common LaPlace Transforms		
Name	Time: $y(t)$ ($t > 0$)	LaPlace: $Y(s)$
delta (impulse)	$\delta(t)$	1
unit step	1	$\left(\frac{1}{s}\right)$
unit ramp	t	$\left(\frac{1}{s^2}\right)$
unit parabola	t^2	$\left(\frac{2}{s^3}\right)$
decaying exponential	e^{-bt}	$\left(\frac{1}{s+b}\right)$
	$t e^{-bt}$	$\left(\frac{1}{(s+b)^2}\right)$
	$t^2 e^{-bt}$	$\left(\frac{2}{(s+b)^3}\right)$
	$2a \cdot e^{-bt} \cos(ct - \theta)u(t)$	$\left(\frac{a \angle \theta}{s+b+jc}\right) + \left(\frac{a \angle -\theta}{s+b-jc}\right)$

Example: Find the impulse response of

$$G(s) = \left(\frac{5}{s+3} \right)$$

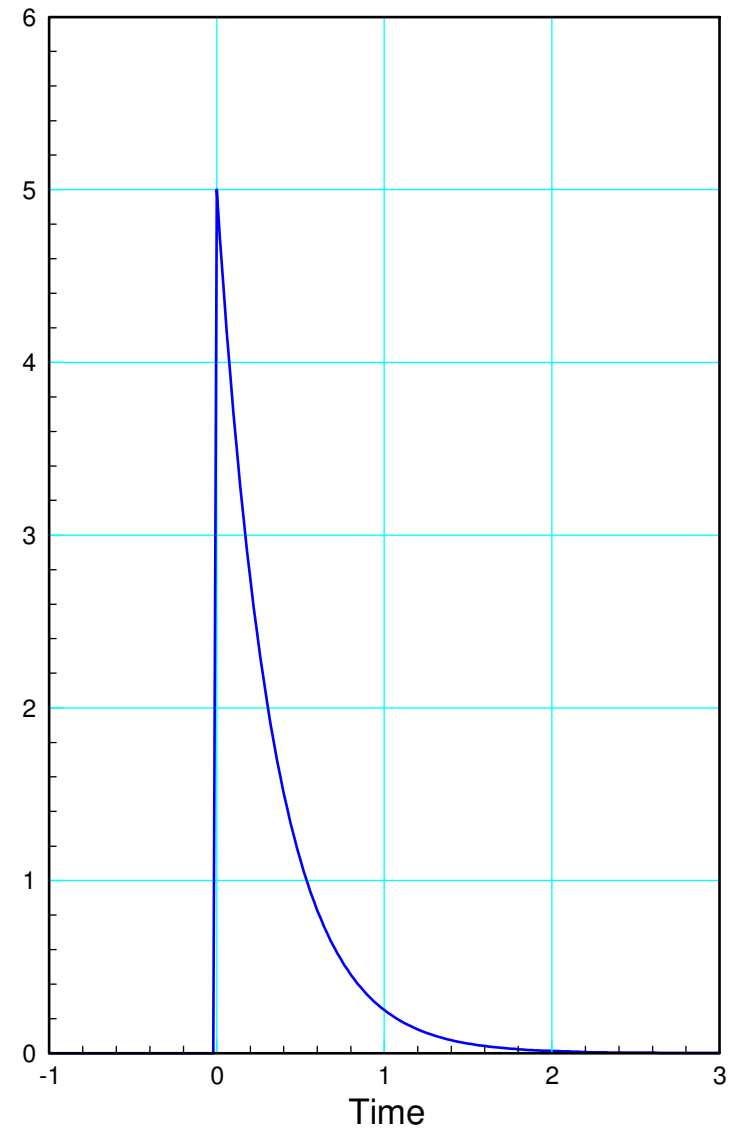
Solution: Translating:

$$Y = \left(\frac{5}{s+3} \right) X$$

$$Y = \left(\frac{5}{s+3} \right) (1)$$

From the above table:

$$y(t) = 5e^{-3t}u(t)$$



Example: Find the step response of

$$G(s) = \left(\frac{5}{s+3} \right)$$

Solution: Translating:

$$Y = \left(\frac{5}{s+3} \right) \left(\frac{1}{s} \right) = \left(\frac{5}{s(s+3)} \right)$$

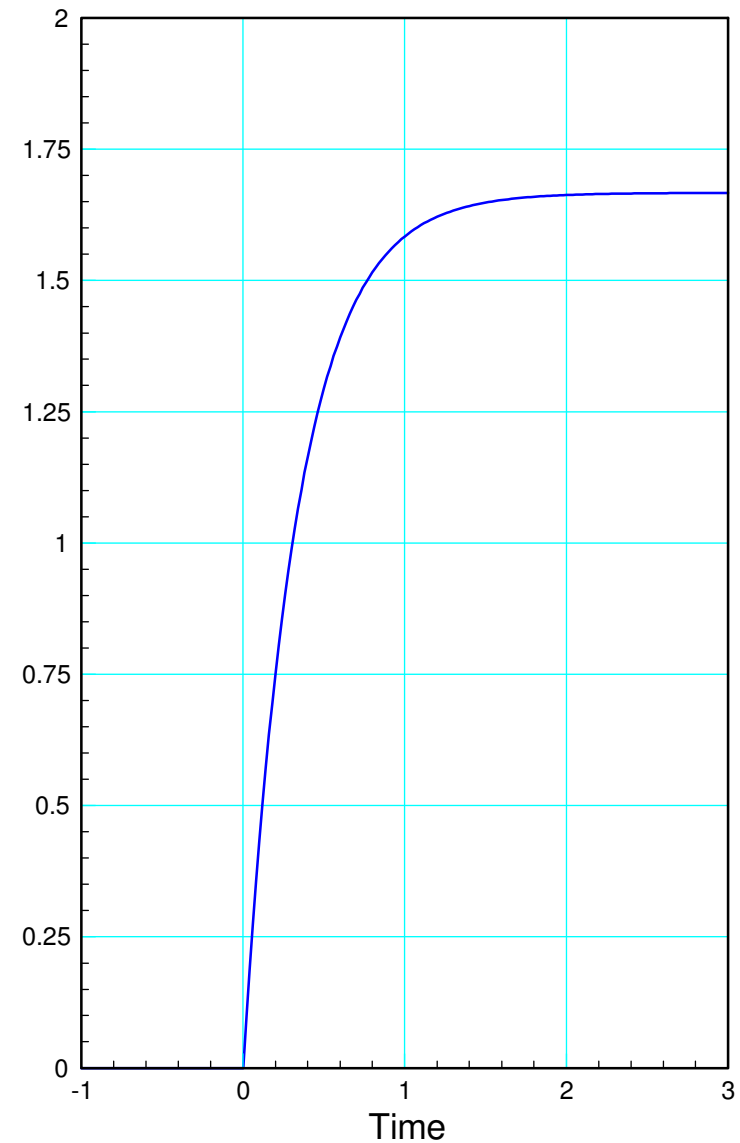
Do a partial fraction expansion

$$Y = \left(\frac{5}{s(s+3)} \right) = \left(\frac{A}{s} \right) + \left(\frac{B}{s+3} \right)$$

$$Y = \left(\frac{5}{s(s+3)} \right) = \left(\frac{5/3}{s} \right) - \left(\frac{5/3}{s+3} \right)$$

Using the above table for each term:

$$y(t) = \left(\frac{5}{3} - \frac{5}{3}e^{-3t} \right) u(t)$$



Repeated Roots

From the table of LaPlace Transforms

$$\left(\frac{(n-1)!}{(s+a)^n} \right) \leftrightarrow t^n e^{-at} u(t)$$

The highest power can be found using partial fractions and the cover-up method

The other terms can be found by using a common denominator along with algebra

Example 1: Find the inverse-LaPlace transform of

$$Y = \left(\frac{4}{s(s+2)^2} \right)$$

You know the answer is in the form of

$$\left(\frac{4}{s(s+2)^2} \right) = \left(\frac{a}{s} \right) + \left(\frac{b}{(s+2)^2} \right) + \left(\frac{c}{s+2} \right)$$

Find a and b using the cover-up method

$$a = \left(\frac{4}{(s+2)^2} \right)_{s=0} = 1$$

$$b = \left(\frac{4}{s} \right)_{s=-2} = -2$$

Find c by using a common denominator and matching terms

$$\left(\frac{4}{s(s+2)^2}\right) = \left(\frac{1}{s}\right) + \left(\frac{-2}{(s+2)^2}\right) + \left(\frac{c}{s+2}\right)$$

$$\left(\frac{4}{s(s+2)^2}\right) = \left(\frac{1}{s}\right)\left(\frac{(s+2)^2}{(s+2)^2}\right) + \left(\frac{-2}{(s+2)^2}\right)\left(\frac{s}{s}\right) + \left(\frac{c}{s+2}\right)\left(\frac{s(s+2)}{s(s+2)}\right)$$

$$4 = (s+2)^2 - 2s + cs(s+2)$$

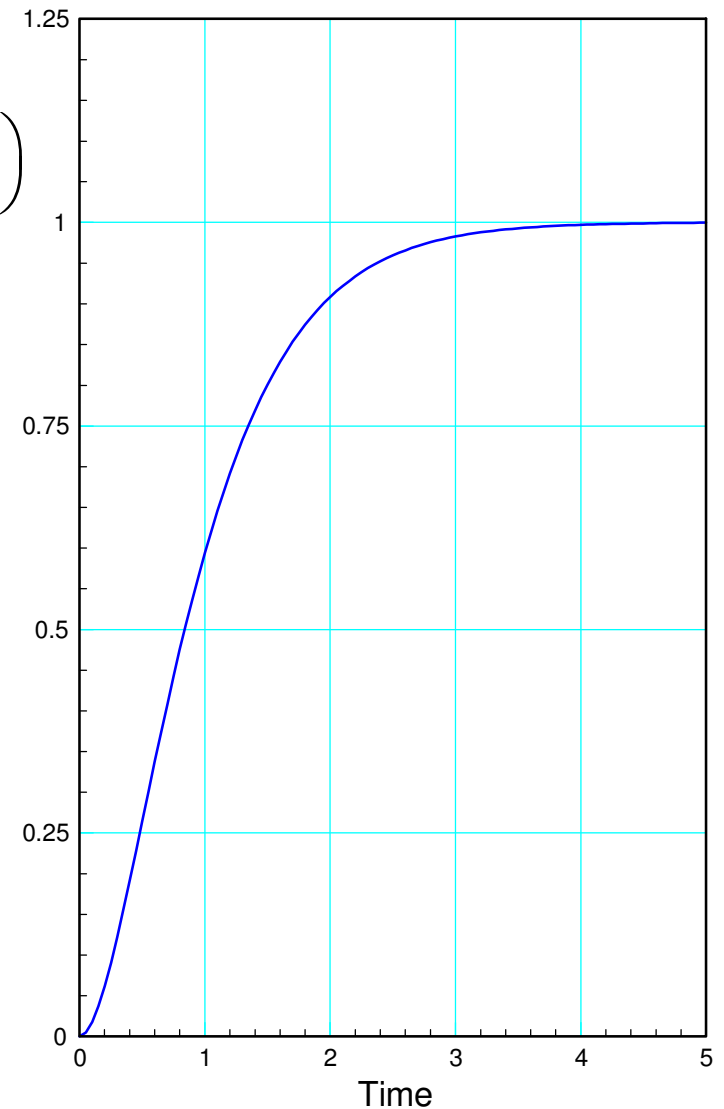
Match the s^2 term

$$0s^2 = s^2 + cs^2 \quad \Rightarrow \quad c = -1$$

So

$$Y = \left(\frac{4}{s(s+2)^2}\right) = \left(\frac{1}{s}\right) + \left(\frac{-2}{(s+2)^2}\right) + \left(\frac{-1}{s+2}\right)$$

$$y(t) = (1 - 2te^{-2t} - e^{-2t})u(t)$$



Example 2: Find $y(t)$

$$Y = \left(\frac{8}{s(s+2)^3} \right)$$

Use partial fractions

$$Y = \left(\frac{a}{s} \right) + \left(\frac{b}{(s+2)^3} \right) + \left(\frac{c}{(s+2)^2} \right) + \left(\frac{d}{s+2} \right)$$

a and b can be found using the cover-up method

$$a = \left(\frac{8}{(s+2)^3} \right)_{s=0} = 1$$

$$b = \left(\frac{8}{s} \right)_{s=-2} = -4$$

To find c and d, place over a common denominator and match terms

$$\left(\frac{8}{s(s+2)^3}\right) = \left(\frac{a}{s}\right) + \left(\frac{b}{(s+2)^3}\right) + \left(\frac{c}{(s+2)^2}\right) + \left(\frac{d}{s+2}\right)$$

$$\left(\frac{8}{s(s+2)^3}\right) = \left(\frac{a}{s}\right)\left(\frac{(s+2)^3}{(s+2)^3}\right) + \left(\frac{b}{(s+2)^3}\right)\left(\frac{s}{s}\right) + \left(\frac{c}{(s+2)^2}\right)\left(\frac{s(s+2)}{s(s+2)}\right) + \left(\frac{d}{s+2}\right)\left(\frac{s(s+2)^2}{s(s+2)^2}\right)$$

Match the numerator

$$8 = a(s+2)^3 + bs + cs(s+2) + ds(s+2)^2$$

s^3 term:

$$0s^3 = as^3 + ds^3$$

$$d = -a = -1$$

s^2 term:

$$0s^2 = 6as^2 + cs^2 + 4ds^2$$

$$0 = 6a + c + 4d$$

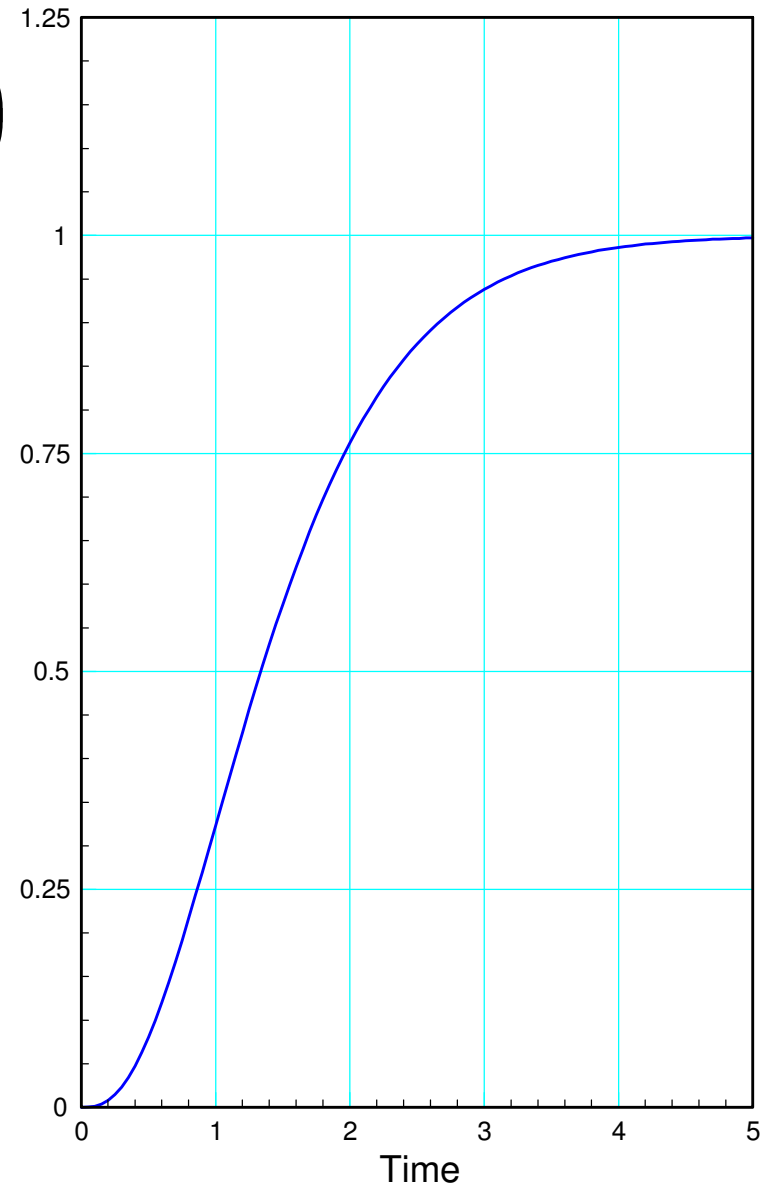
$$c = -2$$

Result:

$$Y = \left(\frac{8}{s(s+2)^3} \right) = \left(\frac{1}{s} \right) + \left(\frac{-4}{(s+2)^3} \right) + \left(\frac{-2}{(s+2)^2} \right) + \left(\frac{-1}{s+2} \right)$$

Take the inverse LaPlace transform

$$y(t) = (1 - 2t^2e^{-2t} - 2te^{-2t} - e^{-2t})u(t)$$



Alternate Method: Change the problem

$$Y = \left(\frac{8}{s(s+2)^3} \right) \approx \left(\frac{8}{s(s+1.99)(s+2.00)(s+2.01)} \right)$$

Now do partial fractions

$$Y = \left(\frac{a}{s} \right) + \left(\frac{b}{s+1.99} \right) + \left(\frac{c}{s+2.00} \right) + \left(\frac{d}{s+2.01} \right)$$

$$Y = \left(\frac{1}{s} \right) + \left(\frac{-20,100.5025}{s+1.99} \right) + \left(\frac{40,000}{s+2} \right) + \left(\frac{-19,900.4975}{s+2.01} \right)$$

Take the inverse LaPlace transform

$$y(t) = (1 - 20,100.5025e^{-1.99t} + 40,000e^{-2t} - 19,900.4975e^{-2.01t})u(t)$$

Comparing Results

- Change the problem gives answers that are good to four decimal places

t	y(t) (exact)	y(t) (approximate)
0	0	0
2.50	0.875347980516919	0.875337136234890
5.00	0.997230604284489	0.997229874022671
7.50	0.999960691551816	0.999960670138687
10.00	0.999999544485050	0.999999544061323

Solving with Complex Roots:

$$\left(\frac{a\angle\theta}{s+b+jc}\right) + \left(\frac{a\angle-\theta}{s+b-jc}\right) \Rightarrow 2a \cdot e^{-bt} \cos(ct - \theta) u(t)$$

Example: Find the $y(t)$ given that

$$Y(s) = G \cdot U = \left(\frac{15}{s^2+2s+10}\right) \cdot \left(\frac{1}{s}\right)$$

Solution: Factoring $Y(s)$

$$Y(s) = \left(\frac{15}{(s)(s+1+j3)(s+1-j3)}\right)$$

Using partial fraction expansion:

$$Y(s) = \left(\frac{1.5}{s}\right) + \left(\frac{0.7906\angle-161.56^\circ}{s+1+j3}\right) + \left(\frac{0.7906\angle161.56^\circ}{s+1-j3}\right)$$

$$y(t) = 1.5 + 1.5812 \cdot e^{-t} \cdot \cos(3t + 161.56^\circ) \quad \text{for } t > 0$$

Properties of LaPlace Transforms

Linearity: $af(t) + bg(t) \Leftrightarrow aF(s) + bG(s)$

Convolution: $f(t) * g(t) \Leftrightarrow F(s) \cdot G(s)$

Differentiation: $\frac{dy}{dt} \Leftrightarrow sY - y(0)$
 $\frac{d^2y}{dt^2} \Leftrightarrow s^2Y - sy(0) - \frac{dy(0)}{dt}$

Integration: $\int_0^t x(\tau) d\tau = \frac{1}{s} X(s)$

Delay: $x(t - T) \Leftrightarrow e^{-sT} X(s)$

Proofs

Linearity:

$$\begin{aligned} L(af(t) + bg(t)) &= \int_{-\infty}^{\infty} (af(t) + bg(t)) \cdot e^{-st} \cdot dt \\ &= \int_{-\infty}^{\infty} (af(t)) \cdot e^{-st} \cdot dt + \int_{-\infty}^{\infty} (bg(t)) \cdot e^{-st} \cdot dt \\ &= a \int_{-\infty}^{\infty} f(t) \cdot e^{-st} \cdot dt + b \int_{-\infty}^{\infty} g(t) \cdot e^{-st} \cdot dt \\ &= aF(s) + bG(s) \end{aligned}$$

Convolution:

$$f(t) * g(t) = \int_{-\infty}^{\infty} f(t - \tau) \cdot g(\tau) \cdot d\tau$$

$$L(f(t) * g(t)) = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(t - \tau) \cdot g(\tau) \cdot d\tau \right) \cdot e^{-st} \cdot dt$$

$$= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(t - \tau) \cdot g(\tau) \cdot e^{-st} \cdot dt \right) \cdot d\tau$$

$$= \left(\int_{-\infty}^{\infty} f(t - \tau) \cdot e^{-st} \cdot dt \right) \cdot \left(\int_{-\infty}^{\infty} g(t) \cdot e^{-st} \cdot dt \right)$$

$$= F(s) \cdot G(s)$$

Differentiation:

$$L\left(\frac{dx}{dt}\right) = \int_{-\infty}^{\infty} \left(\frac{dx(t)}{dt}\right) \cdot e^{-st} \cdot dt$$

Assume causal (zero for $t < 0$)

$$L\left(\frac{dx}{dt}\right) = \int_0^{\infty} \left(\frac{dx(t)}{dt}\right) \cdot e^{-st} \cdot dt$$

Integrate by parts.

$$(ab)' = a' \cdot b + a \cdot b'$$

$$\int a' \cdot b \cdot dt = ab - \int a \cdot b' \cdot dt$$

Let

$$a' = \frac{dx}{dt}$$

$$a = x$$

$$b = e^{-st}$$

then

$$\begin{aligned} L\left(\frac{dx}{dt}\right) &= \int_0^{\infty} \left(\frac{dx(t)}{dt}\right) \cdot e^{-st} \cdot dt \\ &= (x \cdot e^{-st})_0^{\infty} - \int_{-\infty}^{\infty} -s \cdot x(t) \cdot e^{-st} \cdot dt \\ &= -x(0) + s \int_{-\infty}^{\infty} x(t) \cdot e^{-st} \cdot dt \\ &= sX - x(0) \end{aligned}$$

Integration:

$$L\left(\int_0^t x(\tau) \cdot d\tau\right) = \int_{-\infty}^{\infty} \left(\int_0^t x(\tau) \cdot d\tau\right) \cdot e^{-st} \cdot dt$$

Integrate by parts.

$$\int a \cdot b' \cdot dt = ab - \int a' \cdot b \cdot dt$$

Let

$$a = \int_0^t x(\tau) \cdot d\tau$$

$$b' = e^{-st}$$

then

$$a' = x$$

$$b = \frac{-1}{s} e^{-st}$$

$$L\left(\int_0^t x(\tau) \cdot d\tau\right) = \int_{-\infty}^{\infty} \left(\int_0^t x(\tau) \cdot d\tau\right) \cdot e^{-st} \cdot dt$$

$$= \left(\int_0^t x(\tau) \cdot d\tau \cdot \frac{-1}{s} e^{-st} \right)_{-\infty}^{\infty} - \int_{-\infty}^{\infty} x \cdot \frac{-1}{s} e^{-st} \cdot dt$$

Assuming the function vanishes at infinity

$$= \frac{1}{s} \int_{-\infty}^{\infty} x \cdot dt$$

$$= \left(\frac{1}{s} \right) X(s)$$

Time Delay

$$L(x(t - T)) = \int_{-\infty}^{\infty} x(t - T) \cdot e^{-st} \cdot dt$$

Do a change of variable

$$t - T = \tau$$

$$L(x(t - T)) = \int_{-\infty}^{\infty} x(\tau) \cdot e^{-s(\tau+T)} \cdot d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau) \cdot e^{-s\tau} \cdot e^{-sT} \cdot d\tau$$

$$= e^{-sT} \cdot \int_{-\infty}^{\infty} x(\tau) \cdot e^{-s\tau} \cdot d\tau$$

$$= e^{-sT} \cdot X(s)$$

Summary

LaPlace transforms

- Turn differential equations into algebraic equations
- Turn convolution in to multiplication

They are very useful in solving differential equations

- i.e. Analog Signals
- Multiplication is easier than convolution

They will also be useful in Random Processes

- If we ever need to convolve continuous pdf's, LaPlace transforms will turn this into multiplication
- In statistics, LaPlace transforms are termed *moment generating functions*