Uniform Distribution

ECE 341: Random Processes

Lecture #9

note: All lecture notes, homework sets, and solutions are posted on www.BisonAcademy.com

Definitions:

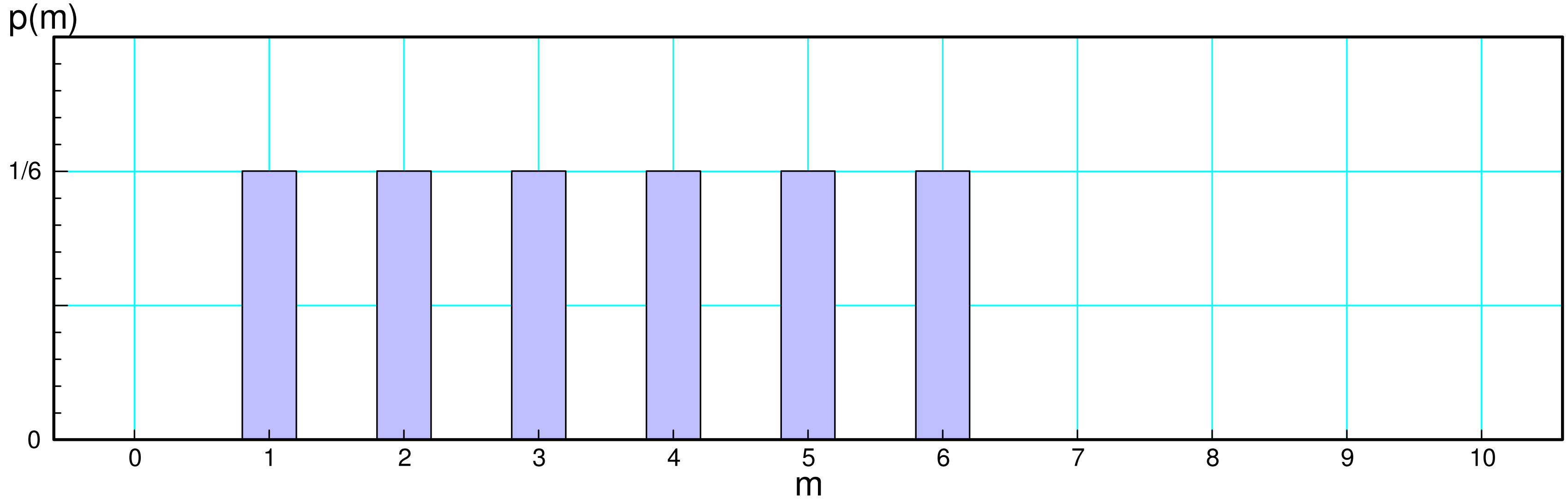
- Uniform Distribution: The probability of each valid outcome is the same.
- Geometric Distribution: The number of Bernoilli trials until you get a success
- Pascal Distribution: The number of Bernolli trials until you get r successes

distribution	description	pdf	mgf	mean	variance
Uniform range = (a,b)	toss an n-sided die	$\left(\frac{1}{1+b-a}\right)\sum_{k=a}^{b}\delta(k-m)$	$\left(\frac{1}{n}\right)\left(\frac{1+z+z^2+\ldots+z^{n-1}}{z^{n-1}}\right)$	$\left(\frac{a+b}{2}\right)$	$\left(\frac{(b+1-a)^2-1}{12}\right)$
Geometric	Bernoulli until 1st success		$\left(\frac{qz}{z-p}\right)$	$\left(\frac{1}{p}\right)$	$\left(\frac{q}{p^2}\right)$
Pascal	Bernoulli until rth success		$\left(\frac{qz}{z-p}\right)^r$	$\left(\frac{r}{p}\right)$	$\left(\frac{qr}{p^2}\right)$

Uniform Distribution:

A Bernoulli trial where

- There are n possible outcomes (rather than just two), and
- All outcomes have the same probability.



pdf for a uniform distribution (6-sided die)

Examples:

- Drawing a card from a deck of cards (each has a probability of 1/52)
- A number coming up in Roulette (1 in 31 in Vegas, 1 in 32 in Atlantic City)
- A number coming up in the lottery (1 in 78,960,960)
- Being selected for jury duty (1 in 15,000. Ten people are selected from a county population of 150,000)

There are some betting schemes which take advantage of processes which are supposed to be uniform but are not. For example, about 15 years ago, someone watched which numbers came up in Roulette in Vegas and found that some numbers were more common that others. He/she (I forget which) won money with this scheme. Now, the roulette wheels are mixed every night (under tight security) and you are not allowed to watch and take notes.

Properties of a Uniform Distribution: (a,b)

pdf: $X(m) = \left(\frac{1}{n}\right) \sum_{k=a}^{b} \delta(k-m)$

mgf $\psi(z) = \left(\frac{1}{b-a}\right)\left(\frac{1}{z^a} + \frac{1}{z^{a+1}} + \ldots + \frac{1}{z^b}\right)$

Mean: $\mu = \left(\frac{a+b}{2}\right)$

Var $\sigma^2 = \left(\frac{(b-a+1)^2-1}{12}\right)$

note: If you add two distributions

- The means add
- The variances add

Dungeons & Dragons

If you're into Dungeons and Dragons, you're use to rolling lots of dice

Different spells do different amounts of damage

- 2d8 means *the sum or rolling two 8-sided dice*

Spell Name	Level	Damage Type	Damage
Frostbite	0	Cold	1d6
Thunderous Smite	1	Thunder	2d6
Mind Whip	2	Psychic	3d6
Thunder Step	3	Thunder	3d10
Ice Storm	4	Bludgeoning + Cold	2d8 + 4d6
Chain Lightning	6	Lightning	10d8
Meteor Swarm	9	Bludgeoning + Fire	20d6 + 20d6

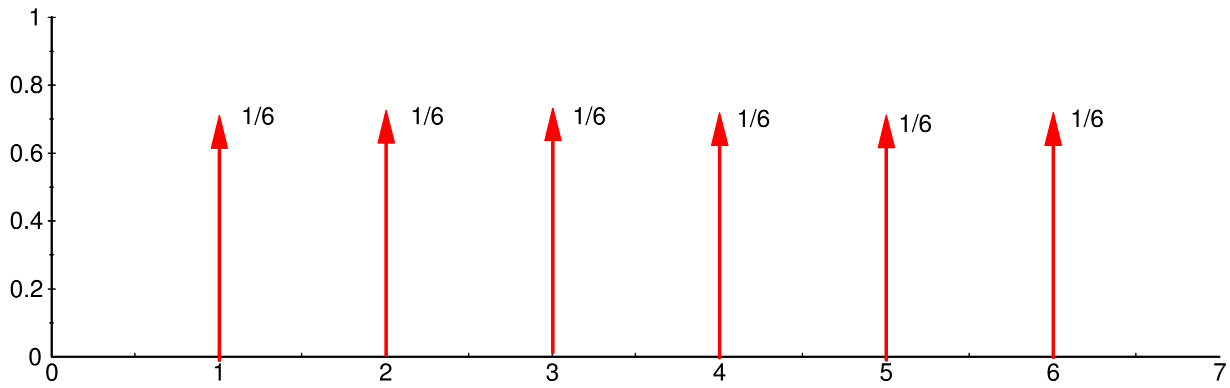
Frostbite: (1d6)

- Druid, sorcerer, warlock, wizard
- Level 0
- 1d6 cold damage

The pdf for 1d6 is:

Die Roll: x	0	1	2	3	4	5	6	7+
p(x)	0	1/6	1/6	1/6	1/6	1/6	1/6	0

This is a delta function at numbers (1..6)



Mean & Variance: 1d6

The mean and variance are:

Mean:

$$\mu = \left(\frac{1+6}{2}\right) = 3.5$$

Variance:

$$\sigma^2 = \left(\frac{(1-3.5)^2+(2-3.5)^2+(3-3.5)^2+(4-3.5)^2+(5-3.5)^2+(6-3.5)}{6}\right) = 2.91167$$

$$\sigma^2 = \left(\frac{(6-1+1)^2-1}{12}\right) = 2.91167$$

Thunderout Smite (2d6)

- Paladin spell
- Level 1
- Weapon does an additional 2d6 of thunder damage

There are several ways to compute the pdf for 2d6

- Monte-Carlo
- Enumeration
- Moment-Generating Functions
- Convolution (preferred)

2d6: Monte-Carlo

- Option #1

Matlab Code	Result: p(x) * 36

```
Damage = zeros(12,1);
for n=1:1e6
  N = sum( ceil(6*rand(2,1)));
  Damage(N) = Damage(N) + 1;
end

Damage = Damage / 1e6;

k = [1:12]';
[k,Damage*36]
```

Damage	M-Carlo
1	0
2	0.9905
3	1.9993
4	3.0073
5	3.9918
6	5.0194
7	6.0026
8	5.0032
9	3.9948
10	2.9965
11	1.9959
12	0.9986

2d6: Enumeration

- Option #2

Matlab Code	Result: p(x) * 36

```
Damage = zeros(12,1);
for d1=1:6
    for d2 = 1:6
        Roll = [d1, d2];
        N = sum(Roll);
        Damage(N) = Damage(N) +
1;
    end
end
k = [1:12]';
[k,Damage]
```

Damage	M-Carlo	Enumeration
1	0	0
2	0.9905	1
3	1.9993	2
4	3.0073	3
5	3.9918	4
6	5.0194	5
7	6.0026	6
8	5.0032	5
9	3.9948	4
10	2.9965	3
11	1.9959	2
12	0.9986	1

2d6: Moment Generating Functions

- Option #3

For 1d6:

$$\psi(z) = \frac{1/6}{z} + \frac{1/6}{z^2} + \frac{1/6}{z^3} + \frac{1/6}{z^4} + \frac{1/6}{z^5} + \frac{1/6}{z^6}$$

For 2d6:

$$\psi(z) = \left(\frac{1/6}{z} + \frac{1/6}{z^2} + \frac{1/6}{z^3} + \frac{1/6}{z^4} + \frac{1/6}{z^5} + \frac{1/6}{z^6}\right)^2$$

$$\left(\frac{1}{6z^6}\right)^2 (1 + z + z^2 + z^3 + z^4 + z^5)^2$$

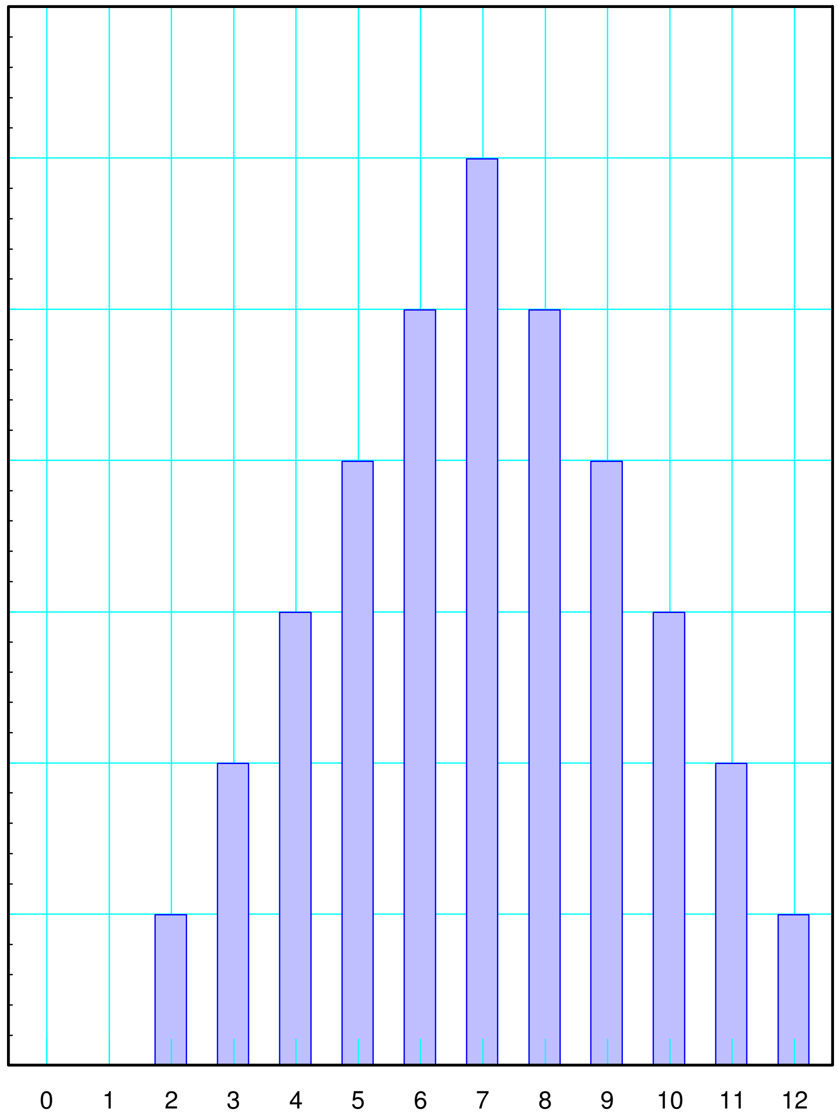
Multiplying polynomials is convolution

- You're stuck using convolution

2d6: Convolution

- 4th option

```
d6 = [0,1,1,1,1,1,1]' / 6;
D  = conv(d6, d6);
k  = [0:length(D)-1]';
bar(k, D)
```



2d6: Mean and Variance

These can be computed using the pdf:

```
x = sum(D .* k)
v = sum(D .* (k - x).^2)

x =      7
v =    5.8333
```

These can be computed by scaling 1d6

1d6

$$\mu = \left(\frac{6+1}{2}\right) = 3.5 \qquad \sigma^2 = \left(\frac{(6-1+1)^2-1}{12}\right) = 2.9167$$

2d6

$$\mu = 2 \cdot 3.5 = 7.000 \qquad \sigma^2 = 2 \cdot 2.9167 = 5.8333$$

Thunder Step (3d10)

- Sorcerer Spell
- Level 3
- 3d10 of thunder damage

You can find the pdf using convolution

- Other methods work

```
d10 = [0,1,1,1,1,1,1,1,1,1,1]' /10;
d10x2 = conv(d10, d10);
d10x3 = conv(d10, d10x2);
D = d10x3;
k  = [0:length(D)-1]';
bar(k, D)
```

Thunder Step (3d10)

Mean and variance come from the pdf:

```
x = sum(D .* k)
v = sum(D .* (k - x).^2)
x =   16.5000
v =   24.7500
```

You can also compute this by scaling

1d10:

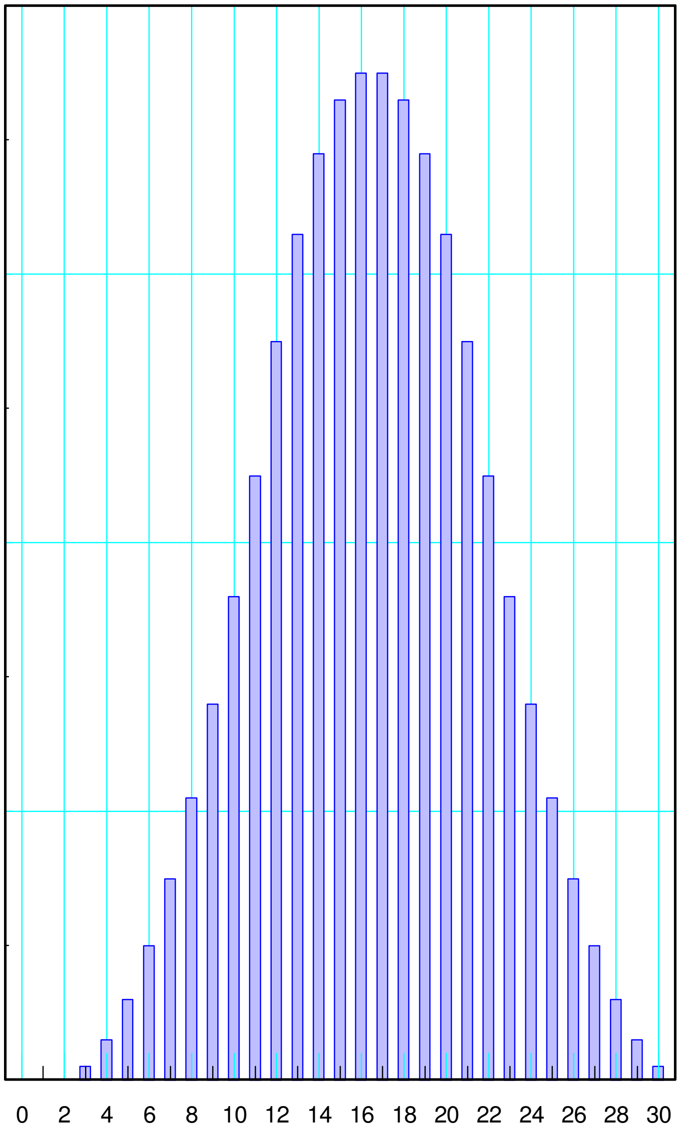
$$\mu = \left(\frac{a+b}{2}\right) = \left(\frac{1+10}{2}\right) = 5.5$$

$$\sigma^2 = \left(\frac{(b-a+1)^2-1}{12}\right) = \left(\frac{10^2-1}{12}\right) = 8.25$$

3d10:

$$\mu = 3 \cdot 5.5 = 16.5$$

$$\sigma^2 = 3 \cdot 8.25 = 24.75$$



Ice Storm: 2d8 + 4d6

- Sorcerer / Wizard / Druid Spell
- Level 4
- 2d8 bludgeoning damage plus 4d6 cold damage

Find the pdf for 2d8 + 4d6

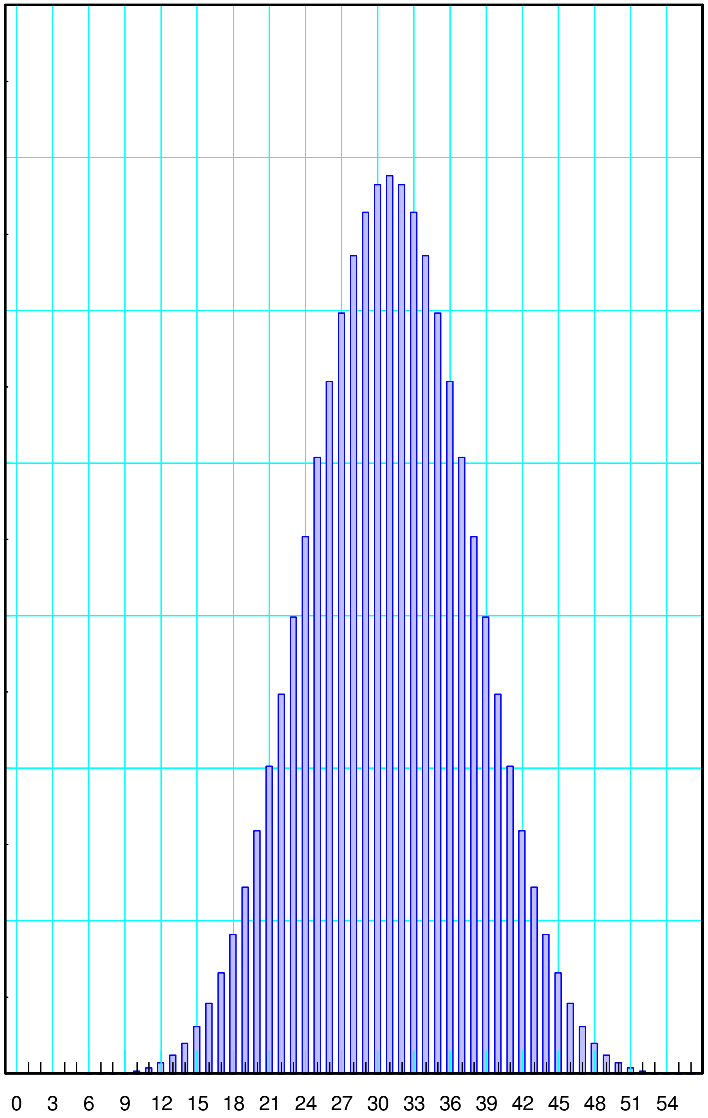
- Convolution works
- Other methods work

```
d6 = [0,1,1,1,1,1,1]'/6;
d6x2 = conv(d6, d6);
d6x4 = conv(d6x2, d6x2);
d8 = [0,1,1,1,1,1,1,1,1]'/8;
d8x2 = conv(d8,d8);
D = conv(d6x4, d8x2);
k  = [0:length(D)-1]';

bar(k,D)
```

Ice Storm: Mean and Variance

Calculate from the pdf:

```
d6 = [0,1,1,1,1,1,1]'/6;
d6x2 = conv(d6, d6);
d6x4 = conv(d6x2, d6x2);
d8 = [0,1,1,1,1,1,1,1,1]'/8;
d8x2 = conv(d8,d8);
D = conv(d6x4, d8x2);
k  = [0:length(D)-1]';
x = sum(D .* k)
v = sum(D .* (k - x).^2)

x =   23.0000
v =   22.1667
```



Ice Storm: Mean and Variance

You can also scale from a single die:

d6:

$$\mu = \left(\frac{1+6}{2}\right) = 3.5 \qquad \sigma^2 = \left(\frac{(6-1+1)^2-1}{12}\right) = 2.91667$$

d8:

$$\mu = \left(\frac{1+8}{2}\right) = 4.5 \qquad \sigma^2 = \left(\frac{(8-1+1)^2-1}{12}\right) = 5.25$$

4d6 + 2d8:

$$\mu = 4 \cdot 3.5 + 2 \cdot 4.5 = 23.00 \quad \sigma^2 = 4 \cdot 2.91667 + 2 \cdot 5.25 = 22.1666$$

Same result

German Tank Problem

Switching gears a bit - here's another problem related to uniform distributions.

Assume you have a population of size N

- Each item is labeled with a unique number 1..N

Assume further, you collect k samples from this population.

- Each item has equal likelihood of being sampled.
- This is a uniform distribution

What is the value of N based upon these k samples?

Origin of the German Tank Problem

The Allies had a problem in WWII

- Sherman medium tanks could not stand up to the German Panther and Tiger tanks
- This led to the nickname "Tommy Cooker"

If the Germans had thousands of these tanks, D-Day would have to be postponed

- Pershing heavy tanks would be needed
- These tanks would need to be built and shipped

If, on the other hand, Germany just had a few hundred, D-Day could go on

- Just not enough German tanks to change the outcome

Sample of Size k

There was data

- Captured German tanks had their production numbers painted on the side
- The gear boxes were also numbered
- The tires were also numbered

Captured tanks provided a sample

- Assume a random sample
- Each tank has equal likelihood of being captured

From this data, how many tanks exist?

Statistical Solution

- N is the population size (highest number)
- m is the largest number from the sampled data
- k is the number of samples

Frequentist Approach:

$$N \approx m + \frac{m}{k} - 1$$

Baysian Approach:

$$N \approx m + \frac{m \ln(2)}{k-1}$$

Baysian Confidence Interval

$$N \approx \mu \pm \sigma$$

$$\mu = (m-1)\left(\frac{k-1}{k-2}\right)$$

$$\sigma^2 = \frac{(k-1)(m-1)(m-k+1)}{(k-3)(k-2)^2}$$

Historical Results:

- https://en.wikipedia.org/wiki/German_tank_problem

The monthly production of German tanks were estimated using

- Statistics,
- Allied Intelligence, and
- Actual production numbers obtained after the war.

Results:

Month	Statistics	Intelligence	German Records
June 1940	169	1,000	122
June 1941	244	1,550	271
Aug 1942	327	1,550	342

Example:

Generate a population of size N

- N is unknown

Shuffle and select 10 at random

Find

- The maximum ID number (m)
- The sample size (k)

```
>> N = round(200*rand) + 50;
>> X = rand(1,N);
>> [a,b] = sort(X);
>> Sample = b(1:10)

Sample =   122    98   173
31    39    21   174    33
151    91

>> m = max(Sample)
m =   174

>> k = length(Sample)
k =    10
```

Example: Estimate the sample size (N)

Frequentist Approach:

```
>> N1 = m + m/k - 1
N1 =  190.4000
```

Baysian Approach:

```
>> N2 = m + m*log(2)/(k-1)
N2 =  187.4008
```

Baysian Confidence Interval

```
>> N3 = (m-1)*(k-1)/(k-2)
N3 =  194.6250

>> s2 = (k-1)*(m-1)*(m-k-1) / ( (k-3)*(k-2)^2 );

>> s = sqrt(s2)
s =   23.8012

>> Upper = N3+s
Upper =  218.4262

>> Lower = N3-s
ans =  170.8238
```

N is actually 213

Frequentist Approach:

```
>> N1 = m + m/k - 1
N1 =  190.4000
```

Baysian Approach:

```
>> N2 = m + m*log(2)/(k-1)
N2 =  187.4008
```

Baysian Confidence Interval

```
>> N3 = (m-1)*(k-1)/(k-2)
N3 =  194.6250

>> s2 = (k-1)*(m-1)*(m-k-1) / ( (k-3)*(k-2)^2 );

>> s = sqrt(s2)
s =   23.8012

>> Upper = N3+s
Upper =  218.4262

>> Lower = N3-s
ans =  170.8238
```

Summary

Uniform distributions have equal probability for all possible outcomes.

- A typical example is the result of rolling an N-sided die.

When you add dice together, you can find the pdf using

- Monte-Carlo
- Enumeration
- Convolution

When adding dice together

- The means add and
- The variance adds

Given k samples from a numbered population of size N

- You can estimate the size of the population using solutions to *The German Tank Problem*