Bernoulli Trials & Binomial Distributions

ECE 341: Random Processes

Lecture #8

note: All lecture notes, homework sets, and solutions are posted on www.BisonAcademy.com

Definitions:

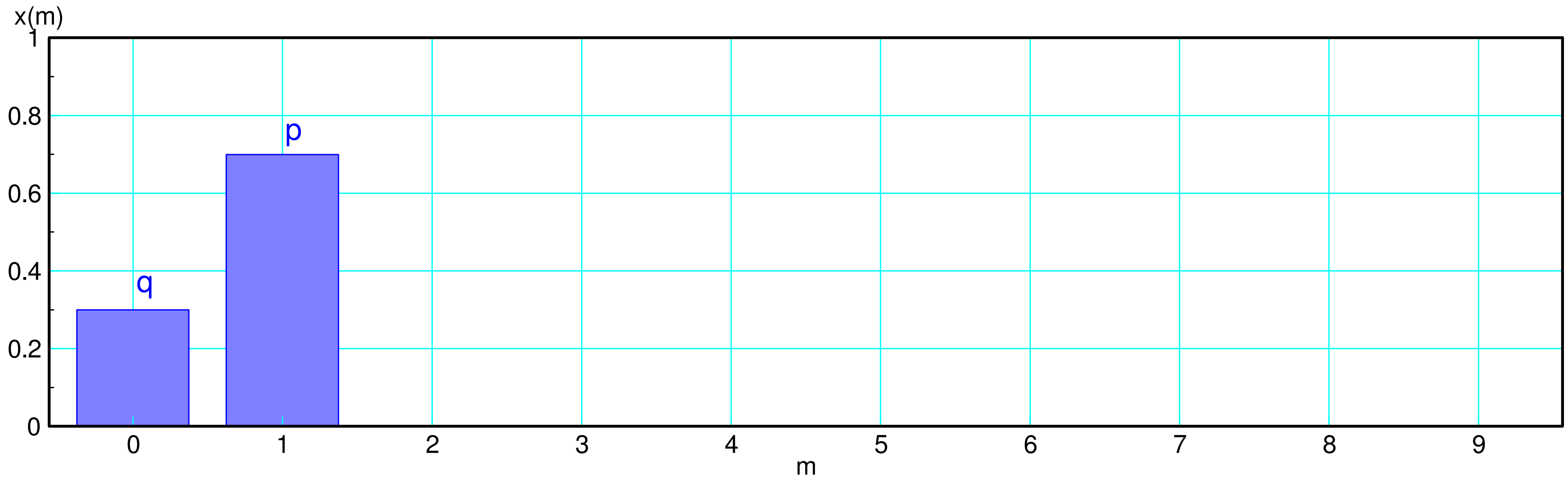
- Bernoulli Trial: A random event whose outcome is true (1) or false (0).
- Binomial Distribution: n Bernoulli trials.
- p The probability of a true (1) outcome (also called a success)
- q 1-p. The probability of a false (0) outcome (also called a failure)
- n The number of trials
- $f(x|n,p)$ The probability density function with sample size n and probability of success p.

distribution	description	pdf	mgf	mean	variance
Bernoulli trial	flip a coin obtain m heads	$p^m q^{1-m}$	$q+\frac{p}{z}$	p	p(1-p)
Binomial	flip n coins obtain m heads	$\binom{n}{m} p^m q^{n-m}$	$\left(q+\frac{p}{z}\right)^n$	np	np(1-p)
Hyper Geometric	Bernoulli trial without replacement	$\frac{\binom{A}{x}\binom{B}{n-x}}{\binom{A+B}{n}}$			

Bernoulli Trial:

A Bernoulli trial is essentially a coin flip:

- There are only two possible outcomes (heads or tails, 1 or 0, success or failure)
- The probability of a success is p
- The probability of a failure is q (which must be 1-p for probabilities to add to 1.000)



probability density function for a Bernoulli trial. For illustration, q = 0.3, p = 0.7

With only two possible outcomes, there are only two bars. You can also write this as

$$x(m) = q \cdot \delta(m) + p \cdot \delta(m-1)$$

Note that the mathematics doesn't care what integer *m* represents:

- In ECE 343 Signals and Systems, m represents time
- In ECE 341 Random Processes, m represents the number of successes

The math doesn't care: it's just an integer, m. Likewise, the same math used in Signals and Systems (i.e. the z-transform) applies to Random Processes.

The z-operator is defined as

$$zx(m) \equiv x(m+1)$$

where *zX* is read as *the next value of x*. If multiplying by z moved forward in time, then dividing by z moves backwards in time

$$z^{-1}x(m) \equiv x(m-1)$$

Likewise, the z-transform for a Bernoulli trial (termed the *moment generating function*) is

$$\psi(z) = q + z^{-1}p = \left(\frac{qz+p}{z}\right)$$

Mean

- The mean is the average.
- Your average winnings from a coin toss are

$$\mu = mean$$

$$\mu = \sum k \cdot p(k)$$

$$\mu = (0 \cdot q) + (1 \cdot p)$$

$$\mu = p$$

If you get $1 if you win, you expect to make $p every time you flip the coin.

Variance & Standard Deviation

The variance and standard deviation is a measure of the variability

$$\sigma^2 = \textit{variance}$$

$$\sigma = \textit{standard deviation}$$

These are defined as

$$\sigma^2 = \sum p(k) \cdot (k - \bar{x})^2$$

For a Bernoulli trial

$$\sigma^2 = q(0 - p)^2 + p(1 - p)^2$$

$$\sigma^2 = p(1 - p)$$

Bernoulli Trials in Matlab:

Matlab has random number functions:

rand	*generate a random number in the interval of 0..1*
rand(10,1)	*generate a matrix of 10 random numbers over the range of 0..1*
randn	*generate a random number with a Gaussian distribution*

For a Bernoulli trial, use the *rand* function

- randn uses a standard normal distribution - covered later in this course

Flip a coin 5 times with the probability of a heads being 0.7.

- First, generate 5 random numbers in the interval (0,1):

```
X = rand(5,1)

    0.8147
    0.9058
    0.1270
    0.9134
    0.6324
```

- Next, convert this to a binary (1 or 0) number

```
Coin = 1 * (rand(5,1) < 0.7)

     0
     0
     1
     0
     1
```

Flip 100 coins

- Find the mean and standard deviation

Solution: Flip a coin 100 times

```
Coin = 1 * (rand(100,1) < 0.7);
x = mean(Coin)

x =   0.5900      vs. 0.700 expected

s = std(Coin)

s =   0.4943      vs 0.4528 expected
```

Flip a coin 1 million times,

```
Coin = 1 * (rand(1e6,1) < 0.7);
x = mean(Coin)

x =    0.6997        vs. 0.7000 expected

s = std(Coin)

s =    0.4584        vs. 0.4528 expected
```

That is essentially the definition of probability: as the number of coin flips goes to infinity, the number of successes divide by the number of trials approaches p

$$\lim_{trials \to \infty} \left(\frac{\text{\# successes}}{\text{\# trials}} \right) = p$$

Sidelight: What is the probability of the Vikings winning their home opener?

- *This is not a repeatable event*
- *Probability does not properly apply*

Actual Question: What is the betting line on the Vikings winning their home opener?

- *Ideally, the losers pay the winners*
- *The house wins on ties*

Binomial Distribution

- The binomial distribution is the sum of n Bernoulli trials.
- The probability of getting m heads with n trials is

$$X(m) = \binom{n}{m} p^m q^{n-m}$$

There are several ways to get this result.

Enumeration

Question: What is the probability of getting m heads in 2 trials?

Answer 1: Enumerate all possibilities and probabilities:

result	probability
1 1	p * p
1 0	p * q
0 1	q * p
0 0	q * q

or

m	X(m)
2	p^2
1	2pq
0	q^2

Problem: What is the probability of flipping m heads in 3 trials (n=3)

Enumerating all combinations (with 1 meaning a heads (success))

n	combinations	probability
3	111	p^3
2	110, 101, 011	$3p^2q$
1	001, 010, 001	$3pq^2$
0	000	q^3

Convolution

The probability density function for a Bernoulli trial is

$$x(m) = q \cdot \delta(m) + p \cdot \delta(m-1)$$

Another way to write this is

$$x(m) = \sum_k x(k) \cdot \delta(m-k)$$

This works since

$$\delta(k) = \begin{cases} 1 & k = 0 \\ 0 & otherwise \end{cases}$$

Likewise, the delta function pick out only those values of x(k) when k = m, giving

$$x(m) = x(m)$$

That seems kind of silly, but it helps when dealing with two coin tosses. If you are tossing two coins, the resulting pdf is

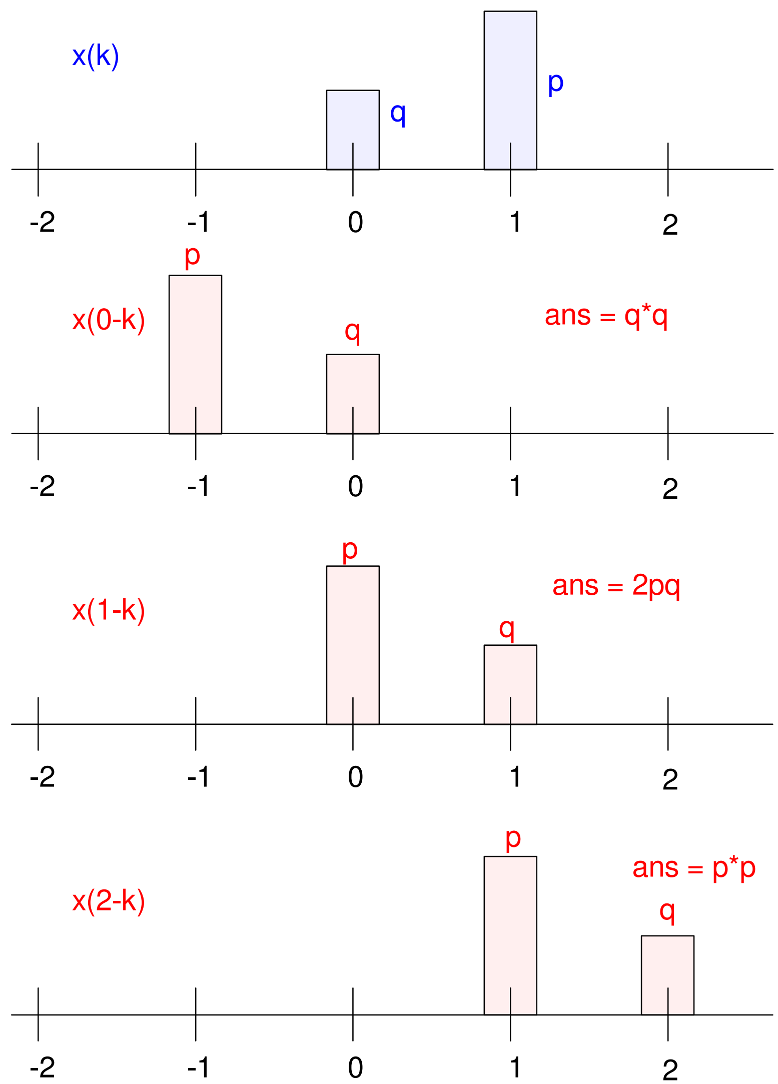
$$y(m) = \sum_k x(k)x(m-k)$$

or

$$y(m) = x(m) ** x(m)$$

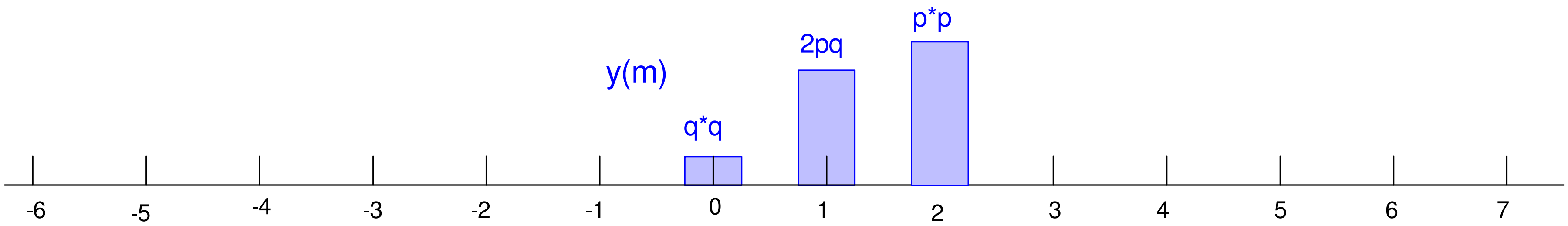
where ** denotes convolution. Graphically, what you are doing is

- Flip x(m) to get x(0-m)
- Multiply x(k) with x(-k) and sum the results. This gives $y(0) = q^2$



Result

$$y(m) = q^2 \cdot \delta(m) + 2qp \cdot \delta(m-1) + p^2 \cdot \delta(m-2)$$



In Matlab, this is the convolution function

```
x = [0.3,0.7]

        x(0)         x(1)
x =    0.3000     0.7000

y = conv(x,x)

         y(0)         y(1)          y(2)
y =    0.0900     0.4200     0.4900
```

z-Transform

A property of the z-transform is

- Convolution in the time domain (probability domain here) is multiplication in the z-domain

This is essentially because multiplying polynomials is convolution.

Example: Find the product of

$$C = AB = (2x^2 + 3x + 4)(5x^3 + 6x^2 + 7x + 8)$$

- Solution using convolution: Flip B(x) and start shifting. The C(0) = 32

B(k)	0	0	0	**8**	7	6	5	0	0	0
A(-k)	0	2	3	**4**	0	0	0	0	0	0
A*B	0	0	0	**32**	0	0	0	0	0	0

Shift A by one and multiply. C(1) = 24 + 28 = 52

B(k)	0	0	0	**8**	**7**	6	5	0	0	0
A(1-k)	0	0	2	**3**	**4**	0	0	0	0	0
A*B	0	0	0	**24**	**28**	0	0	0	0	0

Shift A by one and multiply. C(2) = 16 + 21 + 24 = 61

B(k)	0	0	0	**8**	**7**	**6**	5	0	0	0
A(2-k)	0	0	0	**2**	**3**	**4**	0	0	0	0
A*B	0	0	0	**16**	**21**	**24**	0	0	0	0

Shift A by one and multiply. C(3) = 14 + 18 + 20 = 52

B(k)	0	0	0	8	**7**	**6**	**5**	0	0	0
A(3-k)	0	0	0	0	**2**	**3**	**4**	0	0	0
A*B	0	0	0	0	**14**	**18**	**20**	0	0	0

This is the *conv* operation in Matlab:

```
A = [4,3,2];
B = [8,7,6,5];
C = conv(A,B)

       x0     x1      x2     x3     x4     x5
C =    32     52     61     52     27     10
```

$$(4+3x+2x^2)(8+7x+6x^2+5x^3)$$

$$=32+52x+61x^2+52x^3+27x^4+10x^5$$

The z-transform of two Bernoulli trials is thus

$$Y(z) = (q + z^{-1}p)(q + z^{-1}p)$$

$$Y(z) = q^2 + 2pqz^{-1} + p^2z^{-2}$$

Problem: Determine the probability distribution of flipping 6 coins.

Solution 1: Convolve 6 times

```
X = [0.3, 0.7]
X =    0.3000    0.7000
Y2 = conv(X,X)
Y2 =    0.0900    0.4200    0.4900
Y3 = conv(Y2,X)
Y3 =    0.0270    0.1890    0.4410    0.3430
Y4 = conv(Y3,X)
Y4 =    0.0081    0.0756    0.2646    0.4116    0.2401
Y5 = conv(Y4,X)
Y5 =    0.0024    0.0284    0.1323    0.3087    0.3601    0.1681
Y6 = conv(Y5,X)
Y6 =    0.0007    0.0102    0.0595    0.1852    0.3241    0.3025    0.1176
```

Solution 2: The probability of m heads in n tosses is

$$y(m) = \binom{n}{m} p^m q^{n-m}$$

For example, the probability of 4 heads is

$$y(4) = \binom{6}{4} (0.7)^4 (0.3)^2$$

$$y(4) = 0.324135$$

which matches the Matlab results

```
Y6 = conv(Y5,X)
Y6 =    0.0007    0.0102    0.0595    0.1852    0.3241    0.3025    0.1176
```

Pascal's Triangle

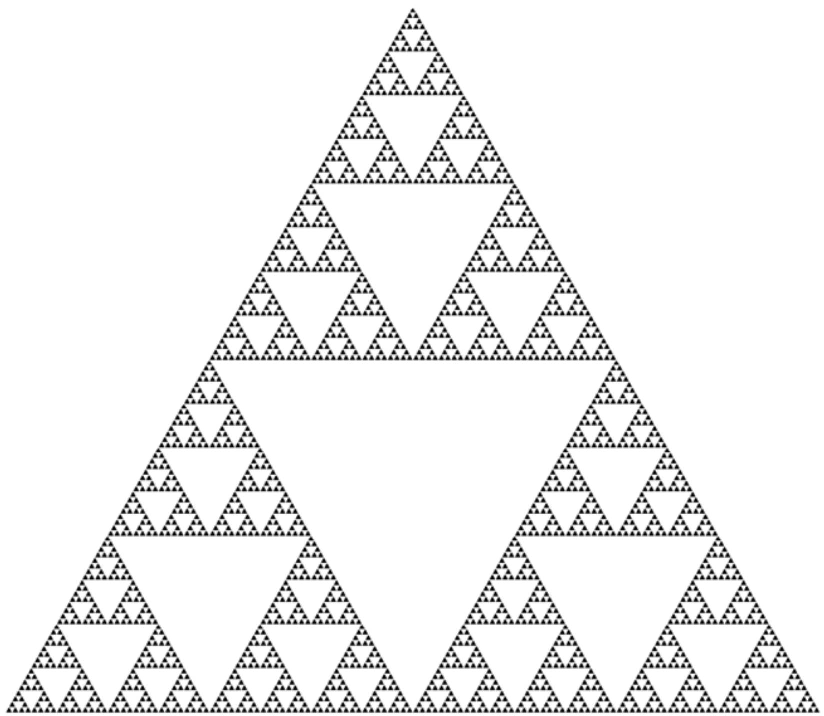
A kind of neat pattern for the combination $\binom{n}{m}$ is as follows:

- Start with the number 1 (or 0 1 0) in row #1
- Offset row #2 by 1/2 a digit. Add the numbers to the left and right of each spot in row #1 to generate row 2.
- Offset row #3 by 1/2 a digit. Add the numbers to the left and right of each spot in row #2 to generate row 3.

The result is as follows:

```
                  1
               1     1
            1     2     1
         1     3     3     1
      1     4     6     4     1
   1     5     10    10    5     1
1     6     15    20    15    6     1
```

Each row is the value of $\binom{n}{m}$ as m goes from zero to n - which is one way of computing combinatorics. If you shade in the odd entries, you get a pretty picture as well: (in the limit becoming Sierpinski Triangle)



SierpinskiTriangle / Pascal's Triangle - source Wikipedia.com

Central Limit Theorem (take 1)

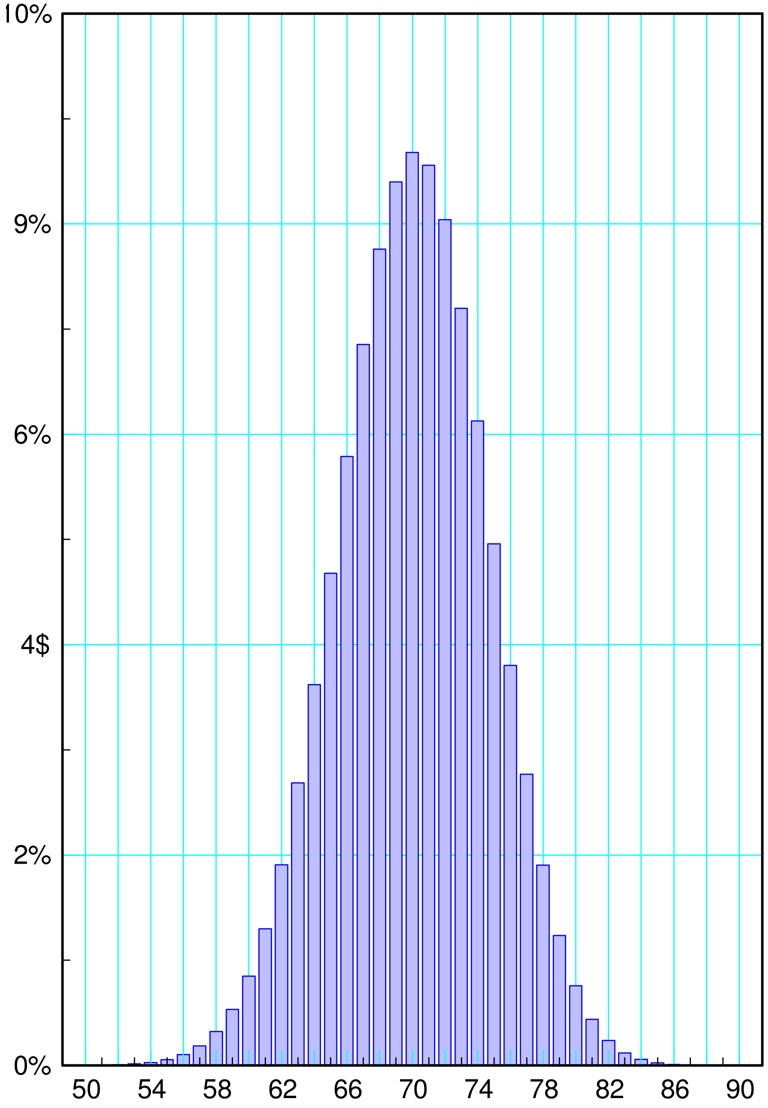
Determine the pdf of flipping 100 coins.

Solution (use Matlab for this):

$$y(m) = \binom{100}{m} p^m q^{100-m}$$

In Matlab

```
Y = zeros(101,1);
for m=0:100
   Y(m+1) = factorial(100) /
   (factorial(m) * factorial(100-m)) *
    0.7^m  * 0.3^(100-m);
end
bar(Y)
```



Note

- Result is a bell-shaped curve (again)

In addition, all probabilities add to one:

```
sum(Y)

ans =   1.0000
```

The mean of a binomial distribution is np (70)

```
sum(Y .* m)

ans =   70.0000
```

The variance is npq (21)

```
sum(Y .* (m - 70).^2)

ans =   21.0000
```

Binomial Distributions with Multiple Outcomes:

Problem: Two people are playing tennis.

- p(A) winning is 60%
- Best of 7 series (first person to win 4 games wins the match)

p(A Winning) = p(A=4) + p(A=5) + p(A=6) + p(A=7)

4 wins: $$f(4) = \binom{7}{4}(0.6)^4(0.4)^3 = 0.2903$$

5 wins: $$f(5) = \binom{7}{5}(0.6)^5(0.4)^2 = 0.2613$$

6 wins $$f(6) = \binom{7}{6}(0.6)^6(0.4)^1 = 0.1306$$

7 wins: $$f(7) = \binom{7}{7}(0.6)^7(0.4)^0 = 0.0280$$

The total is 0.7102.

Win By 2

Problem: Two people are playing tennis.

- p(A) winning is 60%
- First one to be **up** 2 games wins the match

What is the chance that A wins the match?

Solution: This is a *totally* different problem.

- If person A wins followed by a loss, you're back where you started.
- The net result is potentially an infinite series.
- This is a Markov chain (coming later this semester).

Poisson Approximation for a Binomial pdf

Previously lectures used Monte Carlo

- Deal 100,000 hands of poker
- Count number of times each hand appears

These area *actually* binomial distributions

- p = probability of each hand
- n = 100,000
- m = number of each type of hand

$$f(m) = \binom{100,000}{m} p^m (1-p)^{n-m}$$

```
---- 5 Card Stud Poker ----

Straight Flush  = 2
4 of a kind     = 30
Full House      = 141
Flush           = 176
Straight        = 425
3 of a kind     = 2098
2 Pair          = 4635
Pair            = 42276
High Card       = 50217

Elapsed time is 8.045270
seconds.
```

Example: Straight & Royal Flush

The probability of being dealt of of these:

- m = 40 hands
- n = 2,598,960 total poker hands

$$p = \left(\frac{40}{2{,}598{,}960}\right)$$

The probability of x hands

$$f(x) = \binom{100,000}{x} p^x (1-p)^{100{,}000-x}$$

Problem

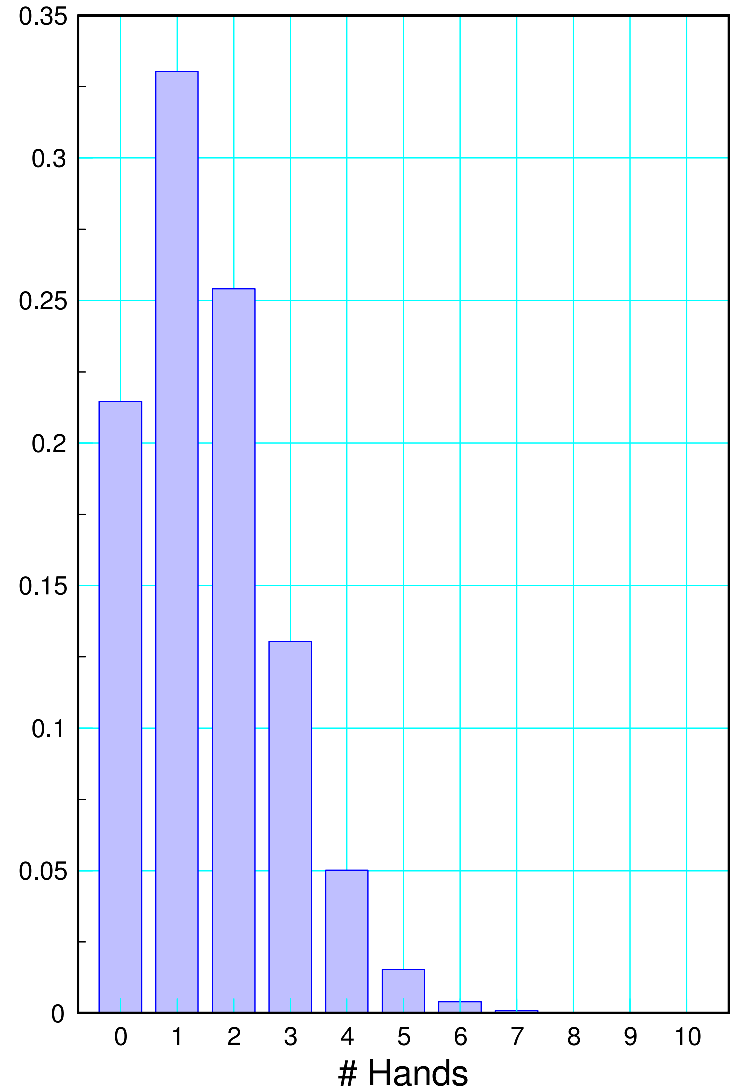
- 100,000! is *way* too large to compute

Poisson Approximation

A Poisson pdf can approximate a binimial:

$$\lambda = np$$

$$f(x) \approx \frac{1}{x!} \cdot \lambda^x \cdot e^{-\lambda}$$



Example:

- Straight or Royal Flushes dealt
- 100,000 hands of poker

$$\lambda = np = (100,000)\left(\frac{40}{2,598,960}\right)$$

$$\lambda = 1.5391$$

$$f(x) \approx \frac{1}{x!} \cdot (1.5391)^x \cdot e^{-1.5391}$$

Marbles: Binomial pdf

Problem: Suppose a box contains A white and B black marbles.

- Each trial, you take one marble out of the bin
- You then replace the marble
- Success if the marble is white

Find the pdf.

Solution: This is a binomial distribution

$$p = \frac{A}{A+B}$$

$$f(x|n,p) = \binom{n}{x} p^x (1-p)^{n-x}$$

Marbles: Hypergeometric pdf

Problem: Suppose a box contains A white and B black marbles.

- Each trial, you take one marble out of the bin
- You then discard the marble
- Success if the marble is white

Find the pdf.

Hypergeometric pdf

The probability of drawing x white marbles in n draws is:

- There are $\binom{A}{x}$ ways of drawing x white balls
- There are $\binom{B}{n-x}$ ways of drawing the remaining n-x black balls
- The total number of ways you can draw n balls is $\binom{A+B}{n}$

So, the pdf for a Hypergeometric distribution is:

$$f(x|A,B,n) = \frac{\binom{A}{x}\binom{B}{n-x}}{\binom{A+B}{n}}$$

Hypergeometric Example

A bowl contains 18 marbles

- 8 white
- 10 black

What is the probability of getting

- 3 white marbles & 2 black marbles
- In 5 draws
- Without replacement?

Solution

$$p = \frac{\binom{8}{3}\binom{10}{2}}{\binom{18}{5}} = \frac{(56)(45)}{8568} = 0.2941$$

Summary

A bernoulli trial has two possible outcomes: Heads (1) and Tails (0)

- The probability of a heads doesn't have to be 1/2

A binomial distribution is the number of successes for n Bernoulli trials

$$pdf(k) = \binom{n}{k} p^k q^{n-k} \qquad\qquad q = 1 - p$$

You can combine Bernoilli trials using

- Convolution, or
- z-Transforms

The resulting pdf quickly becomes a bell-shaped curve (Normal distribution)

- Central Limit Theorem in action

When you combine distributions

- The mean adds, and
- The variance adds