z-Transforms

ECE 341: Random Processes

Lecture #7

note: All lecture notes, homework sets, and solutions are posted on www.BisonAcademy.com

Recap: LaPlace Transforms

LaPlace transforms are a tool which

- Help with the analysis of differential equations, (Math 265)
- Help with the analysis of RLC circuits with analog inputs (ECE 311), and
- Help with the analysis of continuous probability density functions (ECE 341)

LaPlace transforms assume all functions are in the form of

$$y(t) = e^{st}$$

This turns differentiation into multiplication by 's'

$$\frac{dy}{dt} = s \cdot e^{st} = sY$$

'sY' can then be interpreted to mean *the derivative of y*

- LaPlace transforms turn differential equations in to algebraic equations in s.
- The assumption is that algrbra is easier than calculus

LaPlace Transforms and Differential Equations

LaPlace Transforms

- Converts differential equations into algebraic equations
- Turns convolution into multiplication

Example, solve for y(t):

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 10y = 5\frac{dx}{dt} + 10x$$

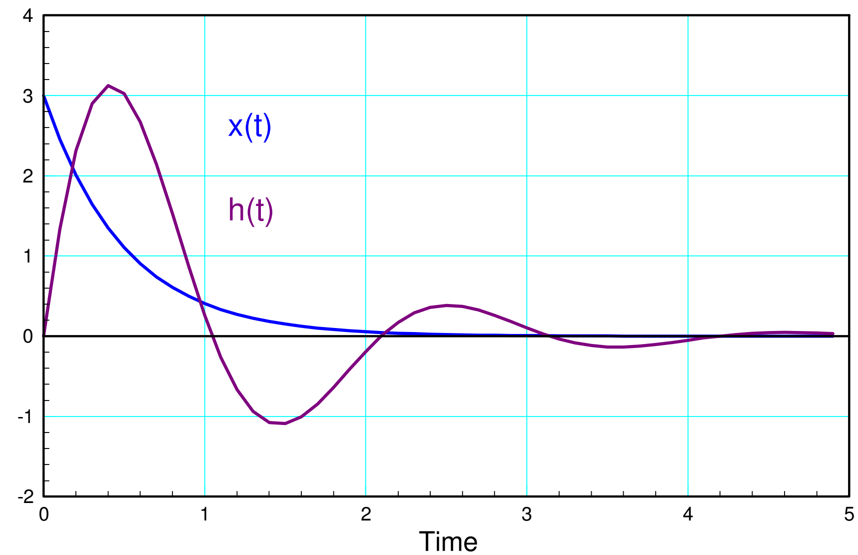
$$x(t) = 3e^{-2t}u(t)$$

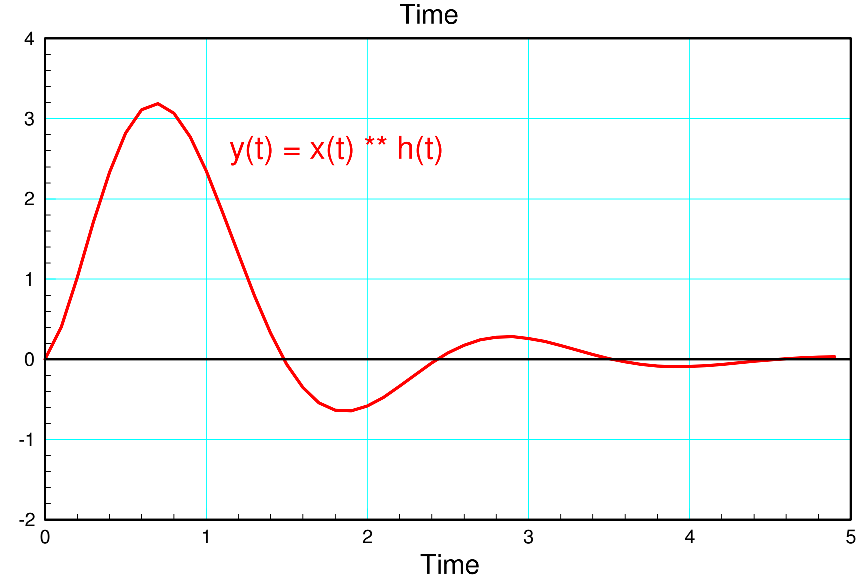
Go to the LaPlace domain

$$s^2Y + 2sY + 10Y = 5sX + 10X$$

$$Y = \left(\frac{5s+10}{s^2+2s+10}\right)X = \left(\frac{5s+10}{s^2+2s+10}\right)\left(\frac{3}{s+2}\right)$$

Use a table to find y(t)





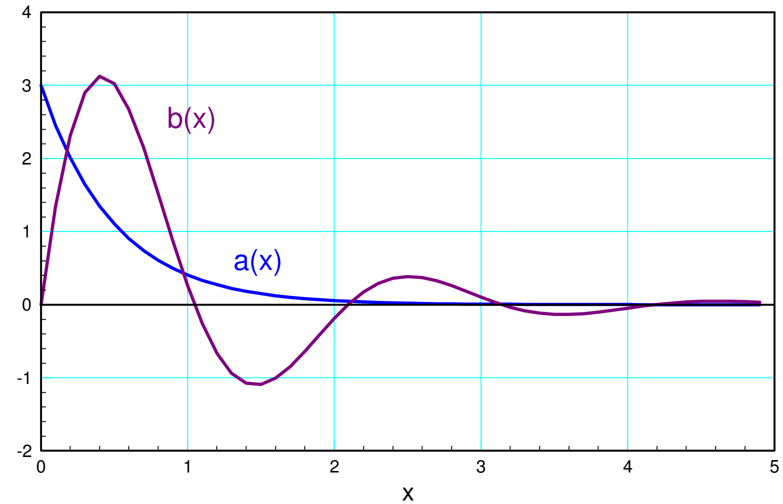
LaPlace Transforms for Random Processes

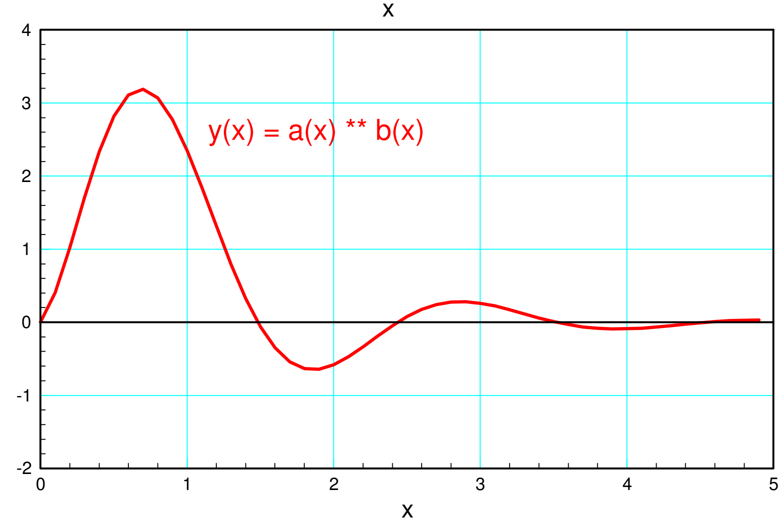
Similarly,

- If we ever need to convolve to continuous probability density functions (pdf),
- Using the LaPlace transform will convert convolution into multiplication

The LaPlace transform of a pdf is termed its *moment generating function*

- Statistics uses a different name, but it's just the LaPlace transform
- Coming soon in ECE 341 when we get to continuous probablity density funcions





z-Transforms

z-Transforms are a tool which

- Help with the analysis of difference equations (ECE 434, ECE 376)
- Help with the analysis of RLC circuits with digital inputs (ECE 461), and
- Help with the analysis of discrete probability density functions (ECE 341)

z-transforms assume all functions are in the form of

$$y(k) = z^k$$

This turns a time advance into multiplication by 'z'

$$y(k+1) = z^{k+1} = z \cdot y(k)$$

zY can then be though of as *the next value of y(k)*

z-Transforms and Difference Equations

z-Transforms

- Convert difference equations into algebraic equations in z, and
- Turn discrete-time convolution into multiplication

For example, find y(k)

$$y(k+2) - 1.9y(k+1) + 0.9y(k) = 0.02(x(k+1) - x(k))$$

Convert to the z-domain

$$z^2 Y - 1.9zY + 0.9Y = 0.02(zX - X)$$

or

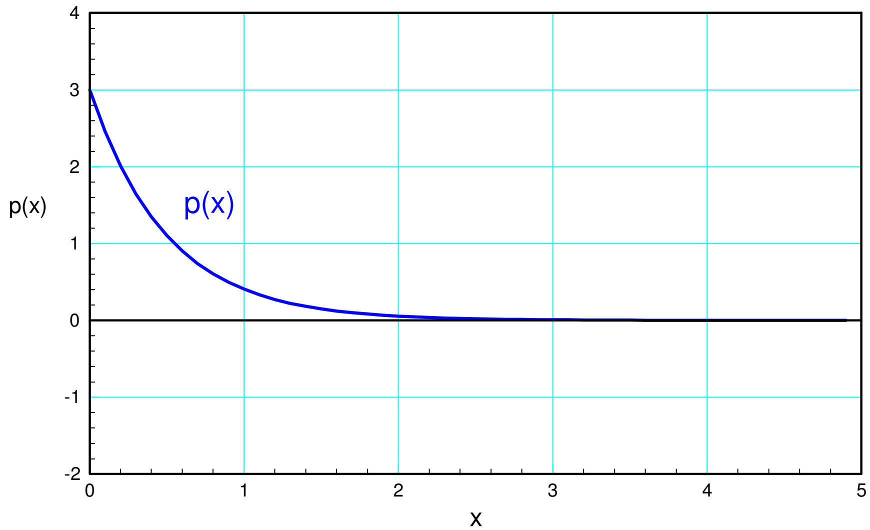
$$Y = \left(\frac{0.02(z-1)}{z^2-1.9z+0.9}\right)X$$

LaPlace and z-Transforms

If you're dealing with continuous functions

- in time
- in probability

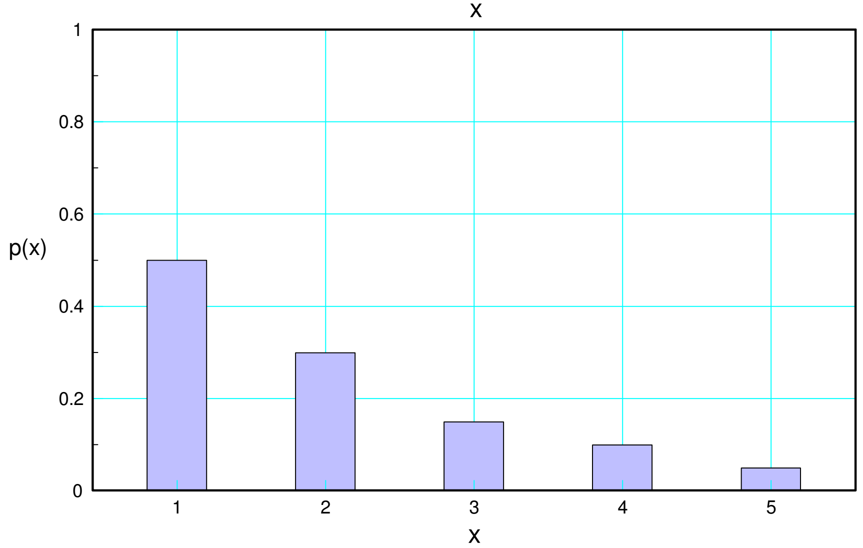
use LaPlace transforms.



If you're dealing with discrete functions

- in time
- in probability

use z-Transforms



z-Transform Properties:

- www.wikipedia.com

The z-transform is defined as

$$X(z) = \sum_{n=-\infty}^{\infty} x_n \cdot z^{-n}$$

Linearity:

$$Z(ax_n + by_n) = aX(z) + bY(z)$$

Proof: The z-transform is

$$Z(ax_n + by_n) = \sum_{n=-\infty}^{\infty} (ax_n + by_n) \cdot z^{-n}$$

$$Z(ax_n + by_n) = \left(a \sum_{n=-\infty}^{\infty} x_n \cdot z^{-n}\right) + \left(b \sum_{n=-\infty}^{\infty} y_n \cdot z^{-n}\right)$$

$$Z(ax_n + by_n) = aX(z) + bY(z)$$

Time Shifting:

$$Z(x_{n-k}) = z^{-k} \cdot X(z)$$

Proof:

$$Z(x_{n-k}) = \sum_{n=-\infty}^{\infty} x_{n-k} \cdot z^{-n}$$

Let m = n-k

$$Z(x_{n-k}) = \sum_{m=-\infty}^{\infty} x_m \cdot z^{-(m+k)}$$

$$Z(x_{n-k}) = \sum_{m=-\infty}^{\infty} x_m \cdot z^{-m} \cdot z^{-k}$$

$$Z(x_{n-k}) = z^{-k} \cdot \left(\sum_{m=-\infty}^{\infty} x_m \cdot z^{-m} \right)$$

$$Z(x_{n-k}) = z^{-k} \cdot X(z)$$

Multiplying by 1/z means delay the signal by one.

Convolution:

$$Z(x_n * *y_n) = X(z) \cdot Y(z)$$

Proof:

$$Z\left(\sum_{k=-\infty}^{\infty} x_k \cdot y_{n-k}\right) = \sum_{n=-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} x_k \cdot y_{n-k}\right) \cdot z^{-n}$$

Change the order of summation:

$$= \sum_{k=-\infty}^{\infty} \left(\sum_{n=-\infty}^{\infty} x_k \cdot y_{n-k}\right) \cdot z^{-n} = \left(\sum_{k=-\infty}^{\infty} x_k \left(\sum_{n=-\infty}^{\infty} y_{n-k}\right)\right) \cdot z^{-n}$$

Let m = n-k

$$= \left(\sum_{k=-\infty}^{\infty} x_k \left(\sum_{n=-\infty}^{\infty} y_m\right)\right) \cdot z^{-(m+k)} = \left(\left(\sum_{k=-\infty}^{\infty} x_k \cdot z^{-k}\right)\left(\sum_{n=-\infty}^{\infty} y_m \cdot z^{-m}\right)\right)$$

$$= X(z) \cdot Y(z)$$

This is a biggie - z-transforms turn convolution into multiplication.

Table of z-Transforms:

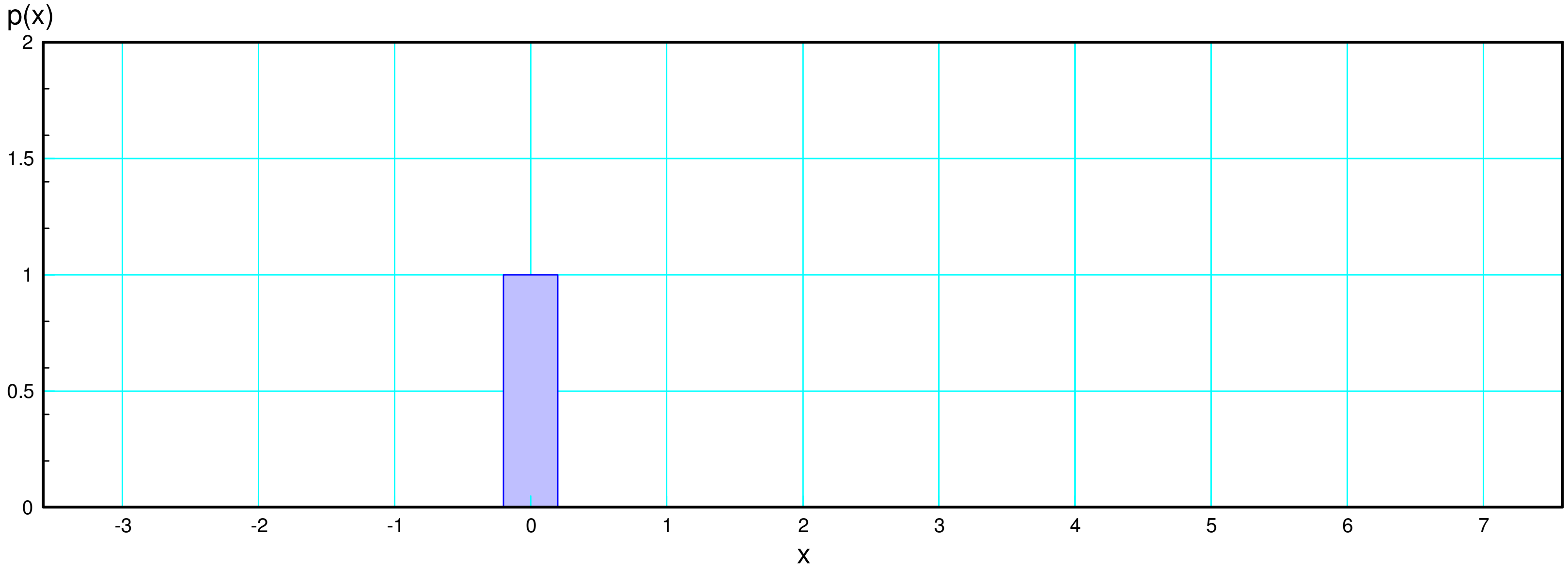
function	y(k) (k > 0)	Y(z)
delta	$\delta(k) = \begin{cases} 1 & k = 0 \\ 0 & otherwise \end{cases}$	1
unit step	$u(k) = 1$	$\left(\frac{z}{z-1}\right)$
ramp	k	$\left(\frac{z}{(z-1)^2}\right)$
parabola	k^2	$\left(\frac{z(z+1)}{(z-1)^3}\right)$
cubic	k^3	$\left(\frac{z(z^2+4z+1)}{(z-1)^4}\right)$
decaying exponential	a^k	$\left(\frac{z}{z-a}\right)$
	$k\ a^k$	$\left(\frac{za}{(z-a)^2}\right)$
	$k^2\ a^k$	$\left(\frac{az(z+a)}{(z-a)^3}\right)$
damped sinewave	$2b \cdot a^k \cdot \cos(k\theta + \phi) \cdot u(k)$	$\left(\frac{(b\angle\phi)z}{z-(a\angle\theta)}\right) + \left(\frac{(b\angle-\phi)z}{z-(a\angle-\theta)}\right)$

Proof: Delta Function. This is sort-of the definition of z-transform:

$$X(z) = \sum_{n=-\infty}^{\infty} x_n \cdot z^{-n} = \ldots + x_0 \cdot z^0 + x_1 \cdot z^1 + x_2 \cdot z^2 + \ldots$$

If x(k) is a delta function:

$$X(z) = 1$$

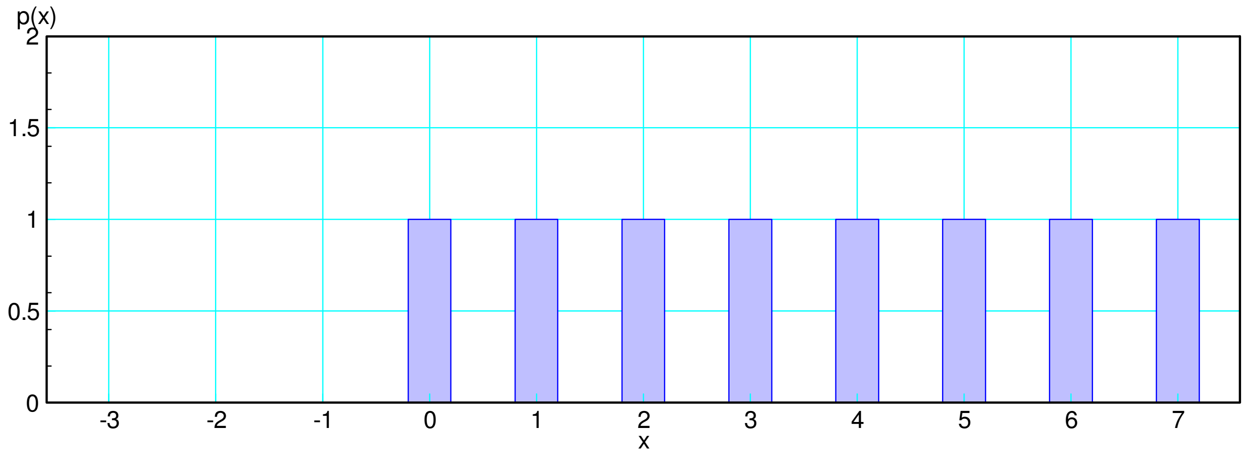


Proof: Unit Step. Using a table:

	z^2	z^1	z^0	z^{-1}	z^{-2}	z^{-3}	z^{-4}
X(z)	0	0	1	1	1	1	1
z^{-1} X(z)	0	0	0	1	1	1	1
subtract							
$\left(1-\frac{1}{z}\right)X(z)$	0	0	1	0	0	0	0

SO

$$X(z) = \frac{1}{\left(1-\frac{1}{z}\right)} = \left(\frac{z}{z-1}\right)$$

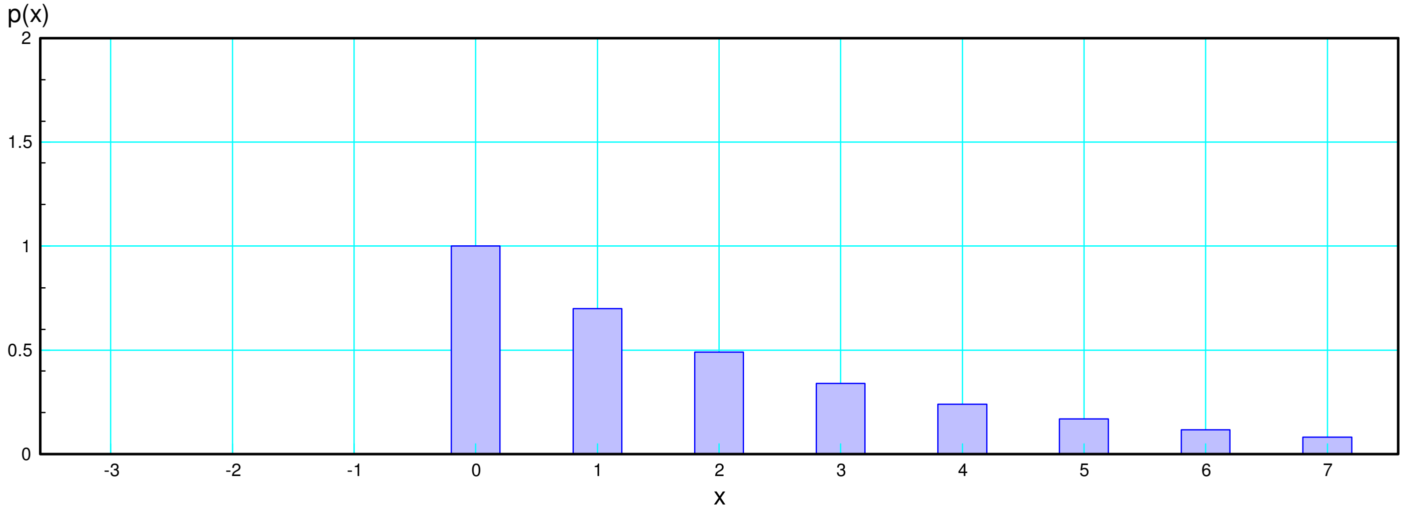


Proof: Decaying Exponential. Using a table:

	z^2	z^1	z^0	z^{-1}	z^{-2}	z^{-3}	z^{-4}
X(z)	0	0	1	a	a^2	a^3	a^4
$a * z^{-1} X(z)$	0	0	0	a	a^2	a^3	a^4
subtract							
$(1-\frac{a}{z})X(z)$	0	0	1	0	0	0	0

so

$$X(z) = \left(\frac{1}{1-\frac{a}{z}}\right) = \left(\frac{z}{z-a}\right)$$



Solving Functions in the z-Domain

Problem 1: Find the step response of

$$Y = \left(\frac{0.2z}{(z-0.9)(z-0.5)}\right)X$$

i) Replace X(z) with the z-transform of a step

$$Y = \left(\frac{0.2z}{(z-0.9)(z-0.5)}\right)\left(\frac{z}{z-1}\right)$$

ii) Use partial fractions (pull out a z - we'll need this)

$$Y = \left(\frac{0.2z}{(z-1)(z-0.9)(z-0.5)}\right)z$$

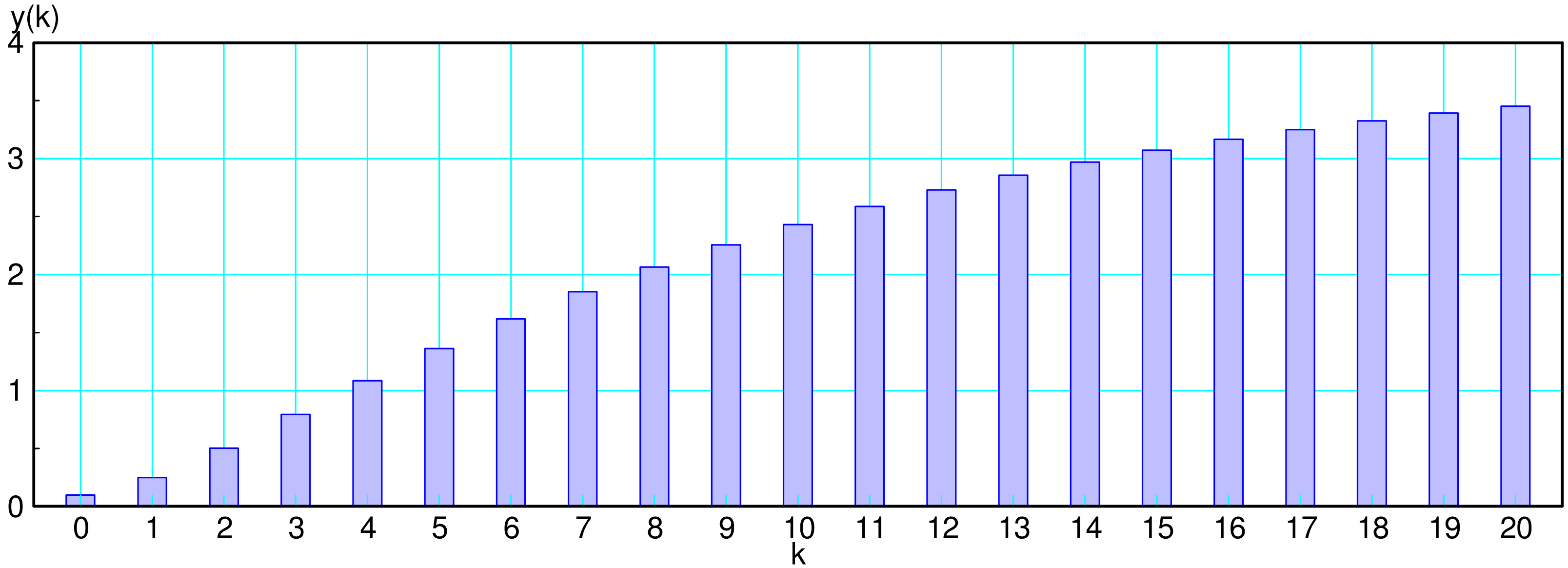
$$Y = \left(\left(\frac{4}{z-1}\right) + \left(\frac{-4.5}{z-0.9}\right) + \left(\frac{0.5}{z-0.5}\right)\right)z$$

Multiply through by z

$$Y = \left(\left(\frac{4z}{z-1} \right) + \left(\frac{-4.5z}{z-0.9} \right) + \left(\frac{0.5z}{z-0.5} \right) \right)$$

iii) Now apply the table entries

$$y(k) = 4 - 4.5 \cdot (0.9)^k + 0.5 \cdot (0.5)^k \qquad k >= 0$$



Problem 2: Find the step response of a system with complex poles:

$$Y = \left(\frac{0.2z}{(z-0.9\angle 10^0)(z-0.9\angle -10^0)}\right) X$$

i) Replace X with its z-transforrm (a unit step)

$$Y = \left(\frac{0.2z}{(z-0.9\angle 10^0)(z-0.9\angle -10^0)}\right)\left(\frac{z}{z-1}\right)$$

ii) Factor our a z and use partial fractions

$$Y = \left(\left(\frac{5.355}{z-1}\right) + \left(\frac{2.98\angle 153.97^0}{z-0.9\angle 10^0}\right) + \left(\frac{2.98\angle -153.97^0}{z-0.9\angle -10^0}\right)\right) z$$

iii) Convert back to time using the table of z-transforms

$$y(k) = 5.355 + 4.859 \cdot (0.9)^k \cdot \cos\left(10^0 \cdot k - 153.97^0\right) \qquad k >= 0$$

Trick if the numberator does not have a z-term

Find the inverse z-transform for

$$Y = \left(\frac{0.2}{(z-1)(z-0.9)(z-0.5)} \right)$$

Multiply by z:

$$zY = \left(\frac{0.2}{(z-1)(z-0.9)(z-0.5)} \right) z$$

Do partial fractions

$$zY = \left(\frac{4}{z-1} - \frac{5}{z-0.9} + \frac{1}{z-0.5} \right) z$$

$$zY = \left(\frac{4z}{z-1} - \frac{5z}{z-0.9} + \frac{z}{z-0.5} \right)$$

Take the inverse z-transform

$$zY = \left(\frac{4z}{z-1} - \frac{5z}{z-0.9} + \frac{z}{z-0.5}\right)$$

$$zy(k) = \left(4 - 5(0.9)^k + (0.5)^k\right)u(k)$$

Divide by z

- time shift: delay by one

$$y(k) = \left(4 - 5(0.9)^{k-1} + (0.5)^{k-1}\right)u(k-1)$$

or equivalently

$$y(k) = \left(4 - 5.555(0.9)^k + 2(0.5)^k\right)u(k-1)$$

Problem 3: Repeated Poles

Find the inverse z-transform for

$$X(z) = \left(\frac{0.2(z+1)}{(z-1)(z-0.9)^2}\right)$$

Repeated Poles

Option 1: With repeated poles, there will be three terms

$$\left(\frac{0.2(z+1)}{(z-1)(z-0.9)^2}\right) = \left(\frac{a}{z-1}\right) + \left(\frac{b}{(z-0.9)^2}\right) + \left(\frac{c}{z-0.9}\right)$$

a and b can be found using the cover-up method

$$a = \left(\frac{0.2(z+1)}{(z-0.9)^2}\right)_{z=1} = 40$$

$$b = \left(\frac{0.2(z+1)}{(z-1)}\right)_{z=0.9} = -3.8$$

Repeated Poles

c can be found by placing over a common denominator

$$\left(\frac{0.2(z+1)}{(z-1)(z-0.9)^2}\right) = \left(\frac{a}{z-1}\right) + \left(\frac{b}{(z-0.9)^2}\right) + \left(\frac{c}{z-0.9}\right)$$

$$\left(\frac{0.2(z+1)}{(z-1)(z-0.9)^2}\right) = \left(\frac{a}{z-1}\right)\left(\frac{(z-0.9)^2}{(z-0.9)^2}\right) + \left(\frac{b}{(z-0.9)^2}\right)\left(\frac{z-1}{z-1}\right) + \left(\frac{c}{z-0.9}\right)\left(\frac{z-1}{z-1}\right)\left(\frac{z-0.9}{z-0.9}\right)$$

Matching the numerator terms

$$0.2(z+1) = a(z-0.9)^2 + b(z-1) + c(z-1)(z-0.9)$$

The z^2 term has to match

$$0 = az^2 + cz^2$$

c = -a

$$\left(\frac{0.2(z+1)}{(z-1)(z-0.9)^2}\right) = \left(\frac{40}{z-1}\right) + \left(\frac{-3.8}{(z-0.9)^2}\right) + \left(\frac{-40}{z-0.9}\right)$$

Now take the inverse-z transform

$$X = \left(\frac{40}{z-1}\right) + \left(\frac{-3.8}{(z-0.9)^2}\right) + \left(\frac{-40}{z-0.9}\right)$$

$$zX = \left(\frac{40z}{z-1}\right) + \left(\frac{-3.8z}{(z-0.9)^2}\right) + \left(\frac{-40z}{z-0.9}\right)$$

$$zx(k) = \left(40 - 3.8k(0.9)^k - 40(0.9)^k\right)u(k)$$

Divide by z

$$x(k) = \left(40 - 3.8(k-1)(0.9)^{k-1} - 40(0.9)^{k-1}\right)u(k-1)$$

Repeated Poles (take 2)

You can also solve by first multiplying both sides by z2 to make the numerator and denominator both 3rd order

$$z^2 X(z) = \left(\frac{0.2z^2(z+1)}{(z-1)(z-0.9)^2}\right)$$

Pull out a z

$$z^2 X = \left(\frac{0.2z(z+1)}{(z-1)(z-0.9)^2}\right) z$$

Do a partial fraction expansion

$$z^2 X = \left(\left(\frac{a}{z-1}\right) + \left(\frac{b}{(z-0.9)^2}\right) + \left(\frac{c}{z-0.9}\right)\right) z$$

time passes....

$$z^2 X = \left(\left(\frac{40}{z-1}\right) + \left(\frac{-3.42}{(z-0.9)^2}\right) + \left(\frac{-39}{z-0.9}\right)\right) z$$

Repeated Poles (cont'd)

$$z^2 X = \left(\frac{40z}{z-1}\right) + \left(\frac{-3.42z}{(z-0.9)^2}\right) + \left(\frac{-39z}{z-0.9}\right)$$

Take the inverse z-transform

$$z^2 x(k) = \left(40 - 3.42k(0.9)^k - 39(0.9)^k\right) u(k)$$

Divide by z^2

$$x(k) = \left(40 - 3.42(k-2)(0.9)^{k-2} - 39(0.9)^{k-2}\right) u(k-2)$$

Both answers are the same

Time Value of Money

Borrow $100,000 at 6% interest for 10 years.

- What are the monthly payments?

Solution:

- x(k) is how much money you owe
- x(k+1) is how much you owe next month (p = monthly payment):

$$x(k+1) = 1.005x(k) - p + X(0) \cdot \delta(k)$$

Take the z-transform (payments start at month #1 rather #0)

$$zX = 1.005X - p\left(\frac{1}{z-1}\right) + X(0)$$

Solve

$$X = \left(\frac{X(0)}{z-1.005}\right) - p\left(\frac{1}{(z-1)(z-1.005)}\right)$$

Using partial fractions

$$X = \left(\frac{X(0)}{z-1.005}\right) + p\left(\left(\frac{200}{z-1}\right) - \left(\frac{200}{z-1.005}\right)\right)$$

$$zX = \left(\frac{X(0)z}{z-1.005}\right) + p\left(\left(\frac{200z}{z-1}\right) - \left(\frac{200z}{z-1.005}\right)\right)$$

Converting back to the time domain

$$zx(k) = 1.005^{k}X(0) - 200p(1.005^{k} - 1)u(k)$$

$$x(k) = 1.005^{k-1}X(0) - 200p(1.005^{k-1} - 1)u(k-1)$$

After 10 years (k=120 payments), x(k) should be zero

- You make 120 payments
- At month 120, your balance is zero, meaning your loan is paid off
- Your monthly payments are \$1117.02

$$x(120) = 0 = \$181,034 - 162.069p$$

$$\boxed{p = \$1117.02}$$

Summary

z-Transforms are similar to LaPlace transforms, but they deal with

- Discrete-time systems
- Discrete probability funcitons.

When dealing with difference equations or discrete-time events, z-transforms will be useful.

In Signals and Systems, X(z) represents a signal

- Its value at z = 1 can be anything

In Random Processes, X(z) represents a probabilty density function

- 0 <= x(k) <= 1 (probabilities can't be negative and must sum to one)