Conditional Probability

5-Card Draw

ECE 341 Random Processes

Lecture #5

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5-Card Draw

Each player is dealt 5 cards

- 1st round of betting

Each player can dicard 0-5 cards

- Draw the same number of cards
- Results in a 5-card hand
- 2nd round of betting

Players reveal their hands

- Best hand wins

What are the odds of each hand?

Enumeration is now very difficult

- Up to 10 cards drawn per person
- $\binom{52}{10} = 15,829,924,220$ different combinations

Further complicated by the number of cards drawn

- If you have a straight, a flush, or a full house, you'll draw zero cards
- If you have 3 of a kind, you might draw two cards
- If you have a pair, you might draw three cards
- If you have junk, you might draw five cards

This makes computations very difficult

- Use conditional probabilities

Conditional Probabilities (review)

$$p(A) = p(A|B)p(B) + p(A|C)p(C) + \ldots$$

Example:

- Player A rolls one die
- Player B rolls two dice and takes the maximum
- The highest number wins. A wins on ties.

What is the probability that A wins?

- 3 dice = 6^3 = 216 permutations
- Treat as a conditional probability

Step 1: Compute the p(B)

- 2 dice take the maximum

By enumeration

B	1	2	3	4	5	6
p(B)	1/36	3/36	5/36	7/36	9/36	11/36

B's Score — 1st Die

2nd Die	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	2	3	4	5	6
3	3	3	3	4	5	6
4	4	4	4	4	5	6
5	5	5	5	5	5	6
6	6	6	6	6	6	6

Step 2: Compute the conditional probabilities

A wins		A's Roll						
		1	2	3	4	5	6	p(B)
B's Roll	1	1	1	1	1	1	1	1/36
	2	0	1	1	1	1	1	3/36
	3	0	0	1	1	1	1	5/36
	4	0	0	0	1	1	1	7/36
	5	0	0	0	0	1	1	9/36
	6	0	0	0	0	0	1	11/36

- p(A) = p(A|B=1)p(B=1) + p(A|B=2)p(B=2) + p(A|B=3)p(B=3) + p(A|B=4)p(B=4) ...
- p(A) = (6/6)(1/36) + (5/6)(3/36) + (4/6)(5/36) + (3/6)(7/36) + (2/6)(9/36) + (1/6)(11/36)
- p(A) = 91/216

A has a 42.13% chance of winning this game

Draw Poker Odds

Treat as a conidtional probability

- p(4 of a kind) = p(4 of a kind | dealt 4 of a kind) p(dealt 4 of a kind)
- + p(4 of a kind | dealt 3 of a kind) p(dealt 3 of a kind)
- + p(4 of a kind | dealt pair) p(dealt a pair)
- + p(4 of a kind | dealt high card) p(dealt high card)

Repeat for all of the hand types

What we know

- From previous lecture
- p(B) for conditional probabilities

Poker Hand	COUNT	Odds Against
Straight Flush	40	64,974
Four of a Kind	624	4,165
Full House	3,744	694.17
Flush	5,108	508.8
Straight	10,200	254.8
Three of a Kind	54,912	47.33
Two Pair	123,552	21.04
One Pair	1,098,240	2.37
High Card	1,302,540	2
Total	2,598,960	

Conditional Probabilities (example)

Assume a fixed rule for the number of cards drawn

- Never attempts to draw to a flush
- Ignores straights (treated as high-card)

Poker hand	# cards drawn
Straight Flush	0
Four of a Kind	0
Full House	0
Flush	0
Straight	4
Three of a Kind	2
Two Pair	1
One Pair	3
High Card	4

Four-of-a-Kind:

Assume there are four ways you can end up with a 4-of-a-kind:

Start with

- A: 4-of-a-kind (and do nothing)
- B: 3-of-a-kind (draw 2 new cards)
- C: Pair (draw 3 new cards)
- D: High-Card hand (draw 4 new cards)

The probability of getting a 4-of-a-kind is

$$p(x) = P(x|A)p(A) + p(x|B)p(B) + p(x|C)p(C) + p(x|D)p(D)$$

A: Starting with 4-of-a-kind

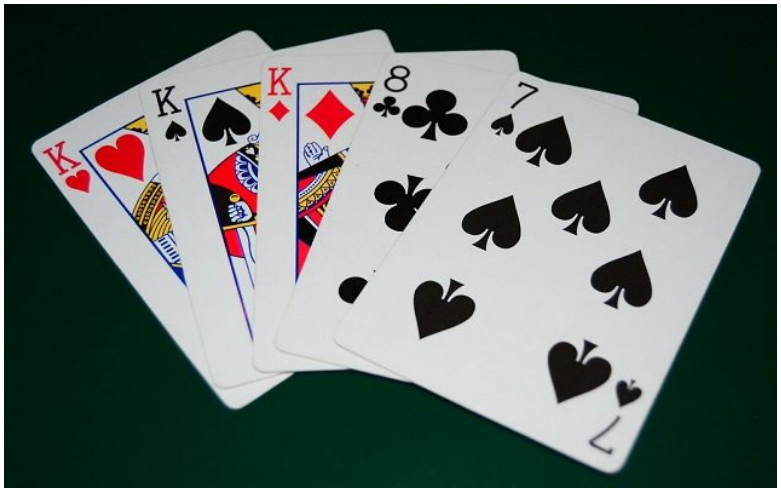
The probability of ending up with a 4-of-a-kind is 1.000

$$p(x|A) = 1.000$$

B: Starting with 3-of-a-kind (draw 2 cards)

The number of ways you can draw 2 cards is

$$M = \binom{47}{2} = 1,081$$



The number of ways you can end up with a 4-of-a-kind is

- Of the one remaining card that matches the cards you keep, choose 1 (1 choose 1)
- Of the 46 remaining cards in the deck, choose 1 (46 choose 1)

$$N = \binom{1}{1}\binom{46}{1} = 46$$

The odds are then

$$p(x|B) = \left(\frac{46}{1081}\right) = \frac{1}{23.5}$$

C: Starting with a pair and you draw 3 new cards

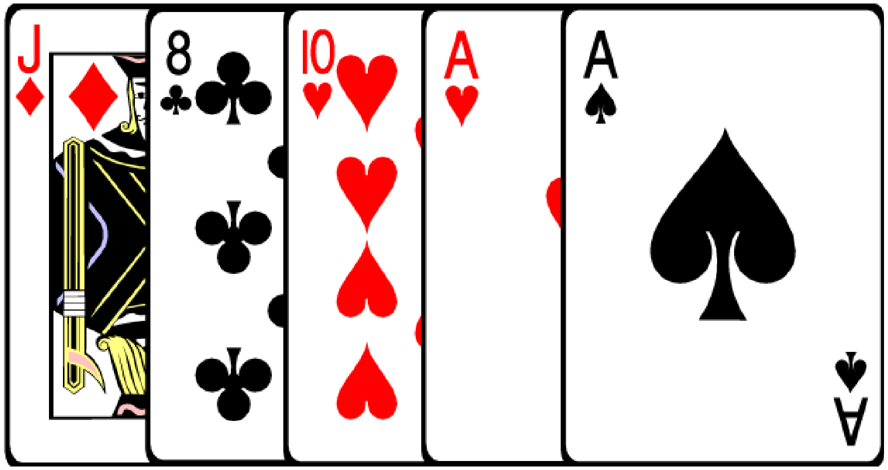
The number of ways you can draw 3 new cards is

$$M = \binom{47}{3} = 16,215$$

To end up with a 4-of-a-kind

- Of the 2 matching cards in the deck, pick two
- Of the remaining 45 cards, pick one

$$N = \binom{2}{2}\binom{45}{1} = 45$$



The odds of getting a four-of-a-kind is then

$$p(x|C) = \left(\frac{45}{16,215}\right) = \left(\frac{1}{360.33}\right)$$

D: High-Card Hand. Draw 4 new cards.

The number of ways to draw 4 cards

$$M = \binom{47}{4} = 178,365$$

The number of ways to get 4 cards that match the one you kept

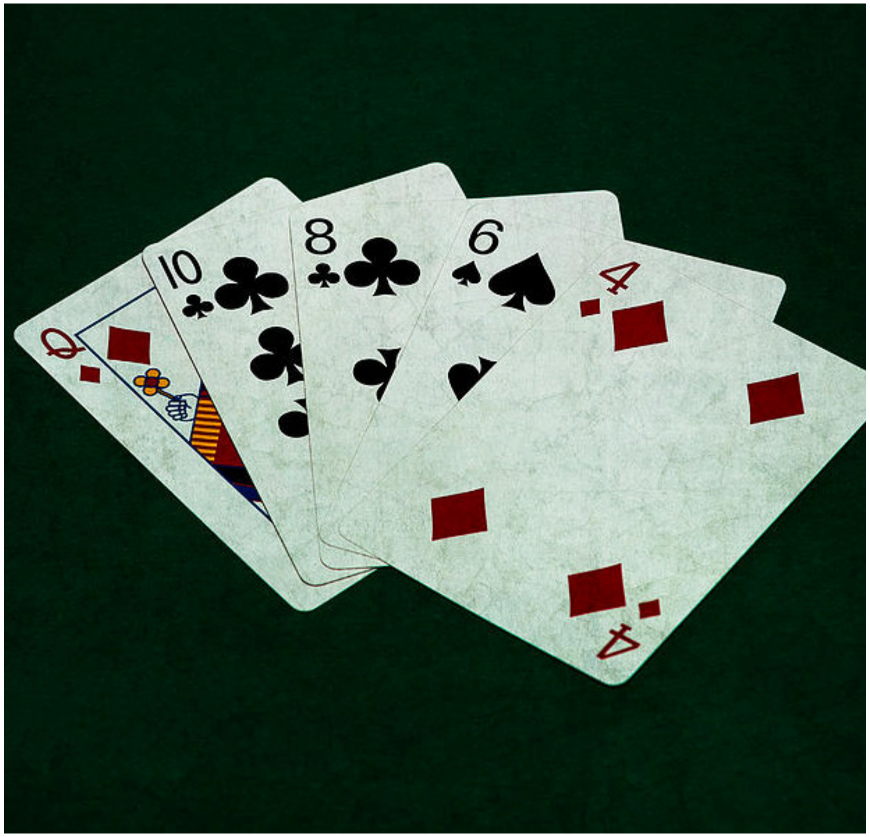
$$N = \binom{3}{3}\binom{44}{1} = 44$$

The number of ways that the 4 cards you drew are all the same

$$N = 8$$

The odds are then

$$p = \frac{44+8}{178,365} = \left(\frac{1}{3430}\right)$$



Result: The probability of getting 4-of-a-kind with 5-card draw is thus

$$p(x) = P(x|A)p(A) + p(x|B)p(B) + p(x|C)p(C) + p(x|D)p(D)$$

$$p(x) = (1.000)\left(\frac{1}{4165}\right) \qquad \textit{dealt 4-of-a-kind}$$

$$+\left(\frac{1}{23.5}\right)\left(\frac{1}{47.33}\right) \qquad \textit{dealt 3-of-a-kind}$$

$$+\left(\frac{1}{360.33}\right)\left(\frac{1}{2.37}\right) \qquad \textit{pair}$$

$$+\left(\frac{1}{3430}\right)\left(\frac{1}{2}\right) \qquad \textit{high-card hand}$$

$$p(x) = 0.002465 = \frac{1}{407.17}$$

Monte-Carlo Simulation

Repeat the previous program for 5-card draw. Add a block of code which

- Sorts the cards based upon their frequency (plus 0.1 * suit so the sort gives unique results)
- Keeps all cards that pair up (frequency > 1)
- Replaces all cards which don't have a pair with the next cards in the deck

Once you draw cards, then recompute how many cards match up (what kind of hand it is)

Matlab Code

Repeat 100,000 times

- Monte-Carlo simulation

Start with a 5-card hand

Check for a flush

- Shows up an NS(1) = 5

Code ignores straights

- not quite correct

```
for games = 1:1e5

   X = rand(1,52);
   [a, Deck] = sort(X);
   Deck = Deck - 1;
   Hand = Deck(1:5);
   Top = 6;

   Value = mod(Hand, 13) + 1;
   Suit = floor(Hand/13) + 1;

% Check for a flush
   NS = zeros(4,1);
   for i=1:4
      NS(i) = sum(Suit == i);
      end

   NS = sort(NS, 'descend');
```

Matlab Code (continued)

Sort the hand based upon frequency of each card

F is

- Each card's frequency (1..5)
- Plus each card's value/100

Example:

```
Value =   6   5   6   12   5

F   =  2.06 2.05 2.06 1.12 2.05
```

Sorted hand is then

```
Value = 6   6   5   5   12
```

Sort so most frequent cards are first

```
% Sort by frequency

F = Value / 100;
for i=1:5
   F(i) = F(i) + sum(Value == Value(i))
   end
[a,b] = sort(F, 'descend');
Value = Value(b);
Suit = Suit(b);
Hand = Hand(b);

for i=1:13
   N(i) = sum(Value == i);
   end
N = sort(N, 'descend')
```

Matlab Code (continued)

- Draw based upon type of hand

Flush, 4-of-a-kind, full-house

- Draw no cards

3-of-a-kind

- Draw two

Two-Pair

- Draw one

Pair

- Draw three

High-Card

- Draw four

```
if(NS(1) == 5)
   disp('flush')
elseif(N(1) == 4)
   disp('4 of a kind')
elseif((N(1)==3)*(N(2)==2))
   disp('full house')
elseif((N(1)==3)*(N(2)<2))
   disp('three of a kind')
   Hand(4:5) = Deck(Top:Top+1);
   Top = Top + 2;
elseif((N(1)==2)*(N(2)==2))
   disp('two pair')
   Hand(5) = Deck(Top);
   Top = Top + 1;
elseif((N(1)==2)*(N(2)<2))
   disp('pair')
   Hand(3:5) = Deck(Top:Top+2);
   Top = Top + 3;
elseif(N(1) < 2)
   disp('High Card')
   Hand(2:5) = Deck(Top:Top+3);
   Top = Top + 4;
end
```

Matlab Code (continued)

Once the draw step is done, determine the hand type

After 100,000 hands, display totals for each hand type

```
   Value = mod(Hand, 13) + 1;
   Suit = floor(Hand/13) + 1;

% Check for a flush
   NS = zeros(4,1);
   for i=1:4
      NS(i) = sum(Suit == i);
      end
   NS = sort(NS, 'descend');

% check for pairs
   for i=1:13
      N(i) = sum(Value == i);
      end
   N = sort(N, 'descend')

if(NS(1)==5) flush=flush+1; end
if(N(1)==4) Pair4=Pair4+1; end
if((N(1)==3)*(N(2)==2)) FH=FH+1; end
if((N(1)==3)*(N(2)<2)) Pair3=Pair3+1; end
if((N(1)==2)*(N(2)==2)) Pair22=Pair22+1;
if((N(1)==2)*(N(2)<2)) Pair2=Pair2+1; end
if(N(1)<2) HighCard=HighCard+1; end
```

Monte-Carlo Simulation Results

Check the program by comparing the results with 0 draws

- Should match previous calculations
- Not exact but close

	0 Draws 5-Card Stud		1-Draw 5-Card Draw	
	Simulation	Calculation	Simulation	Calculation
4 of a kind	22	24	235	243
full house	166	144	1,200	?
flush	196	195	309	?
3 of a kind	2,151	2,112	7,853	?
2-pair	4,791	4,753	13,590	?
1-pair	42,264	42,194	51,863	?
High-Card	50,606	50,578	25,259	?

Number of each type of hand in 100,000 games

Alternate Draw Strategies

Suppose you have

- High-Card, and
- Four cards of a suit

Should you

- Draw four (current strategy), or
- Draw one card and go for a flush?

Solution:

Calculate the probability of ending up with each type of hand

- p(straight-flush) | discard 4 cards
- p(4-of-a-kind) | discard 4 cards
- etc.

vs.

- p(straight-flush) | discard 4 cards
- p(4-of-a-kind) | discard 4 cards
- etc.

Assign a value to each hand type

- Video Poker payout is one option
- Determine which strategy has the highest over value

Alternate Draw Strategies (take 2)

Suppose you have

- High-Card, and
- A run of four cards

Should you

- Draw four (current strategy), or
- Draw one card and go for a straight?

Alternate Draw Strategies (take 3)

Suppose you have

- High-Card, and
- You're missing one card for a straight

Should you

- Draw four (current strategy), or
- Draw one card and go for a straight?
- (termed an inside-straight)

Note: There are better draw strategies

- They're a lot more complicated to analyze
- They're a lot more complicated to code

Poker Variations:

How does adding more draw rounds affect the odds?

- Very complicated to calculate
- Easy to simulate
  - Add multiple draw rounds
  - Implement with a for-loop
- note: This code never tries for a straight or a flush

	0 Draws 5-card stud	1 Draw 5-Card Draw	2 Draws	3 Draws
4 of a kind	22	235	774	1,541
full house	166	1,200	3,647	6,681
flush	196	309	321	372
3 of a kind	2,151	7,853	13,860	18,737
2-pair	4,791	13,590	21,989	27,962
1-pair	42,264	51,863	47,387	38,917
High-Card	50,606	25,259	12,373	6,162

Number of each type of hand in 100,000 games

Texas Hold'Em

Yet another variation of poker is Texas Hold'Em

- Each player is dealt 2 cards
- The dealer then deals out 5 cards on the table, face up
- Your poker hand is the best hand you can form with these seven cards

What are the odds of each type of hand?

Texas Hold-Em

This is a combinatorics problem

- Not a conditional probability
- Solution is left for homework sets

Summary

Adding a draw step changes the odds in poker

Enumeration doesn't work in this case

- Too many hands to go through

Combinatorics with conditional probabilities does work

- List out all possible paths to get to a given hand
- Uses the results from 5-card stud to find p(B)
- Calculate p(A|B)p(B) for each path

Monte-Carlo Simulations also work

- Better draw strategies are more difficult to program
- Also allows you to add more draw steps
  - Use a for-loop