Combinatorics

5-Card Stud Poker

ECE 341 Random Processes

Lecture #4

Please visit Bison Academy for course syllabus, lecture notes, recorded lectures, homework sets, and solutions

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Combinations and Permutations: 5-Card Stud

- Combinatorics: how many ways an event can happen.

Definitions:

- n! "n factorial" n x (n-1) x (n-2) x ... x 2 x 1
- 0! = 1 Just define zero factorial to be one.
- p(x) "the probability of outcome x"
- ${}_nP_m$ "Permutations of n events taken m at a time".
  ${}_nP_m = \frac{n!}{(n-m)!}$
- ${}_nC_m$ "Combinations of n events taken m at a time"
  ${}_nC_m = \frac{n!}{m!\cdot(n-m)!}$
- $\binom{n}{m}$ "n choose m". Another way of writing ${}_nC_m$
- Sample With Replacement: Each sample is of the same population size
- Sample Without Replacement: Each time you sample, the remaining population becomes one smaller

Monte-Carlo vs. Enumeration vs. Combinatorics

Monte-Carlo works

- Play a game N times & count the number of successes
- Results are approximate
- Matlab is a great tool for Monte-Carlo simulations

Enumeration works

- List out all possibilities and count the number of successes
- Answers are exact
- Matlab is a great tool for enumeration
  - Lots of nested for-loops

Combinatorics

- Calculate the number of ways to get a given outcome
- Answers are also exact (but can be tricky to find)

Permutations

- Sample without replacement
- Order matters

Example: Volleyball Team

- How many teams can you make with 20 players?
- Each player is assigned to a position (order matters)

Solution

- 20 ways to pick the fist player
- 19 ways to pick the second...

$$N = 20 \cdot 19 \cdot 18 \cdot 17 \cdot 16 \cdot 15$$

$$N = 27,907,200$$

You can also write this as "20 Pick 6"

$$N = \left(\frac{20!}{(20-6)!}\right) = {}_{20}P_6$$

Combinations

- Sample without replacement
- Order doesn't matter

Example: Volleyball Team

- How many different volleyball teams can you make with 20 players?
- Teams can shuffle positions (order doesn't matter)

Solution

- Each permutation overcounts each time by 6!

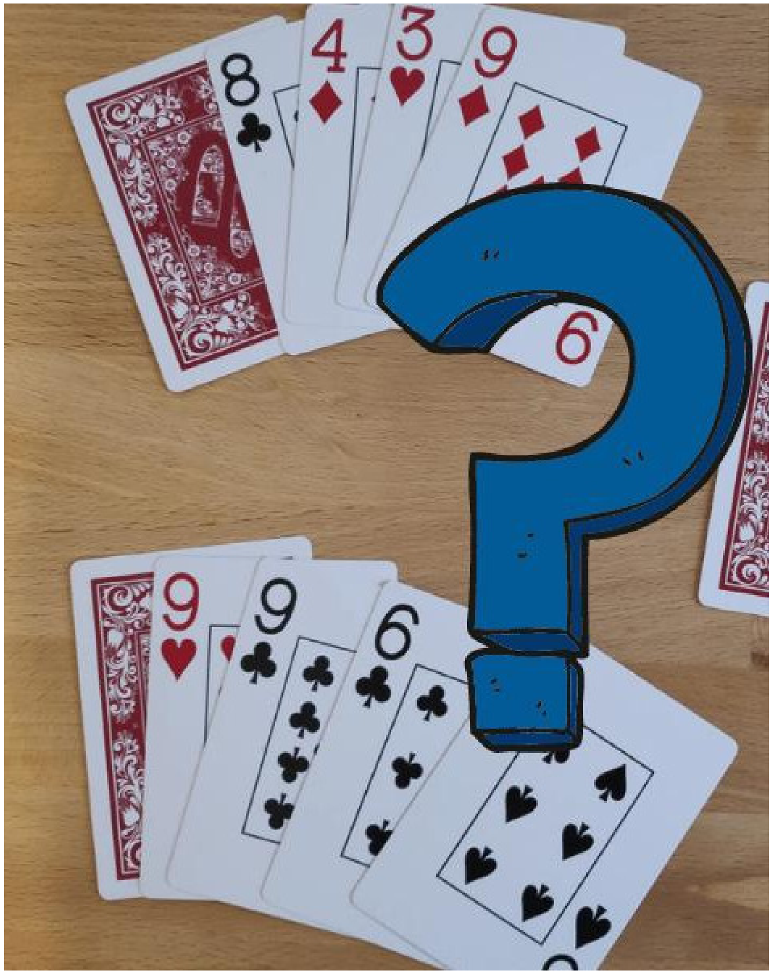
$$N = \frac{_{20}P_6}{6!} = 38,760$$

You can also write this as *20 Choose 6:*

$$N = {}_{20}C_6 = \binom{20}{6} = \left(\frac{20!}{(20-6)!(6)!}\right) = 38,760$$

Poker: 5-Card Stud

- Starting out, a deck of 52 cards is shuffled.
- The first card is played face down so that only the player sees this card.
- The second card is played face up so all players can see it. After two cards are played, bets are made.
- Once betting stops, a third card is played face up and betting starts over again.
- Ditto for the 4th card and 5th card.
- Once betting is finished with 5 cards for each player, the face down card is revealed and the winner is determined.



Rank of Hands

The winning hands (in order) for poker are

- Royal Flush: 10-J-Q-K-A of the same suit
- Straight-Flush: A run of 5 cards in the same suit
- 4 of a kind: Four of your cards have the same value. Ex: J-J-J-J-x
- Full-House: 3 of a kind and a pair. Ex: J-J-J-Q-Q
- Flush: All cards of the same suit
- Straight: A run of 5 cards
- 3 of a kind: Three of your cards match. Ex: J-J-J-x,y
- 2-Pair: Two pairs of cards. J-J-Q-Q-x
- Pair: Two cards match. J-J-x-y-z
- High-Card: Other. No pairs, no straight, no flush.

Probability of Each Type of Hand (take 1)

- Wikipedia

POKER HAND		COUNT	Odds Against
2	Straight Flush	40	64,974
3	Four of a Kind	624	4,165
4	Full House	3,744	694.17
5	Flush	5,108	508.8
6	Straight	10,200	254.8
7	Three of a Kind	54,912	47.33
8	Two Pair	123,552	21.04
9	One Pair	1,098,240	2.37
10	High Card	1,302,540	2
Total		2,598,960	

Probability of Each Type of Hand (take 2)

- Enumeration
- Nested for-loops in Matlab
- Takes about 30 minutes to go through all possible poker hands (2.5 million)

POKER HAND		COUNT	Odds Against
2	Straight Flush	40	64,974
3	Four of a Kind	624	4,165
4	Full House	3,744	694.17
5	Flush	5,108	508.8
6	Straight	10,200	254.8
7	Three of a Kind	54,912	47.33
8	Two Pair	123,552	21.04
9	One Pair	1,098,240	2.37
10	High Card	1,302,540	2
Total		2,598,960	

Probability of Each Type of Hand (take 3)

- Combinatorics

Compute the number of ways to get each type of hand

- Answers should match enumeration (exact answers)
- Takes lots less time
- Kind of tricky

Factorials:

How many ways are there to shuffle a deck of 52 cards?

- Select the first card from 52 possibilities
- The second card from 51 possibilities
- The third card from 50 possibilities.
- etc.

The total number of ways a deck can be shuffled is

$$N = 52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 \cdot ...2 \cdot 1$$

This is 52 factorial, written as

$$N = 52!$$

Most calculators have a factorial key

$$N = 8.0658 \cdot 10^{67}$$

Note that this is too many combinations for even a computer to run through.

Permutations: (Order Matters)

The number of ways a given hand can play out is..

- Select the first card from 52 possibilities
- The second card from 51 possibilities
- The third card from 50 possibilities.

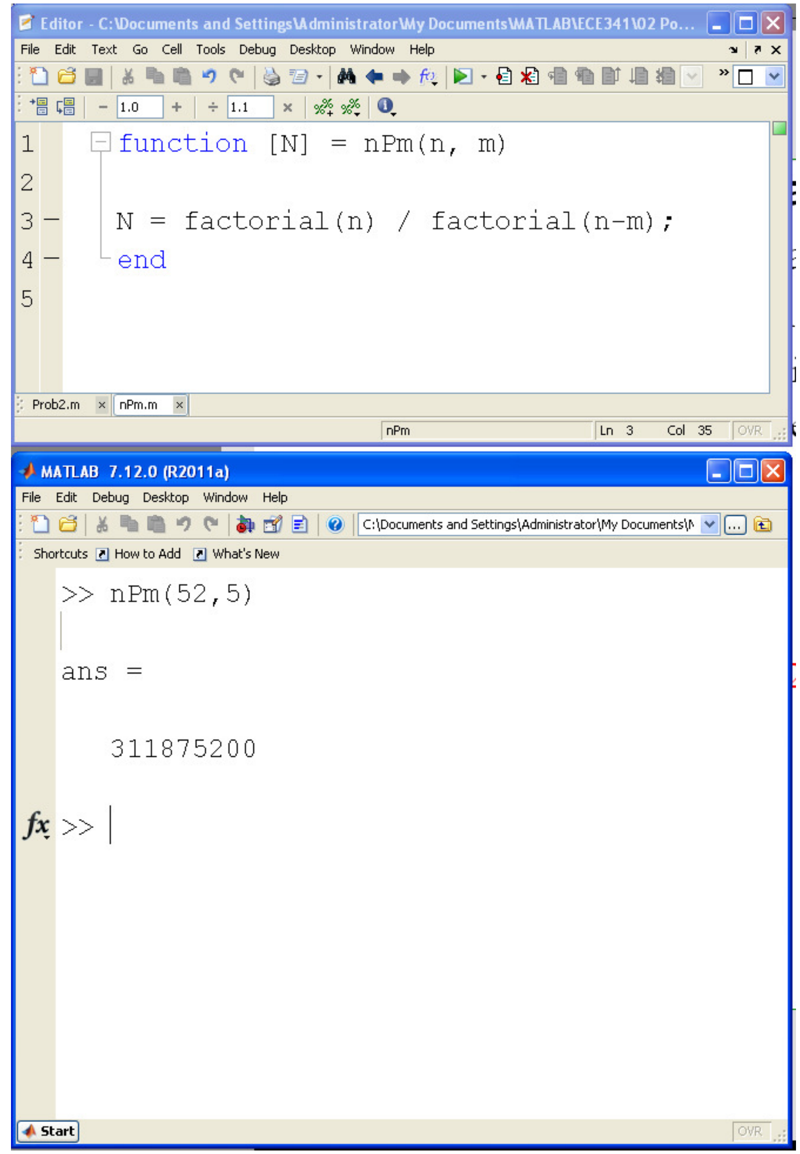
giving

$$N = 52 \cdot 51 \cdot 50 \cdot 49 \cdot 48$$

Another way to write this is

$$N = {}_{52}P_5 = \left(\frac{52!}{(52-5)!} \right) = 311,875,200$$

There are 311,875,20 different ways a given hand can play out.



Combinations

- order doesn't matter

When determining who won the hand, the order of the cards doesn't matter.

- Select the first card from 5 possibilities
- The second card from 4 possibilities
- The third card from 3 possibilities.

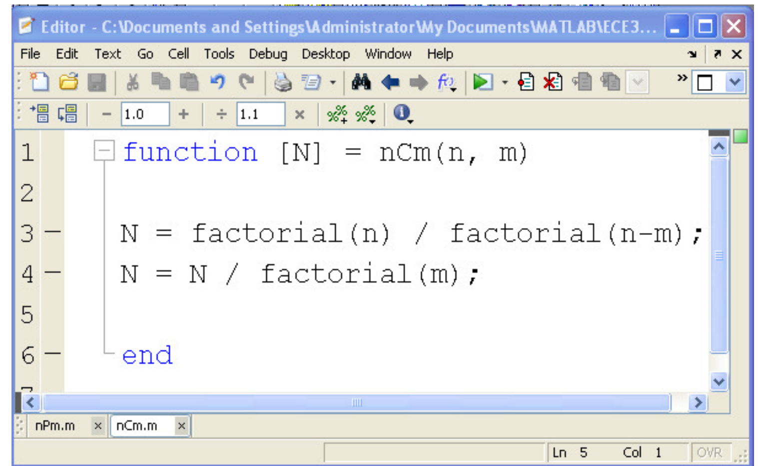
or

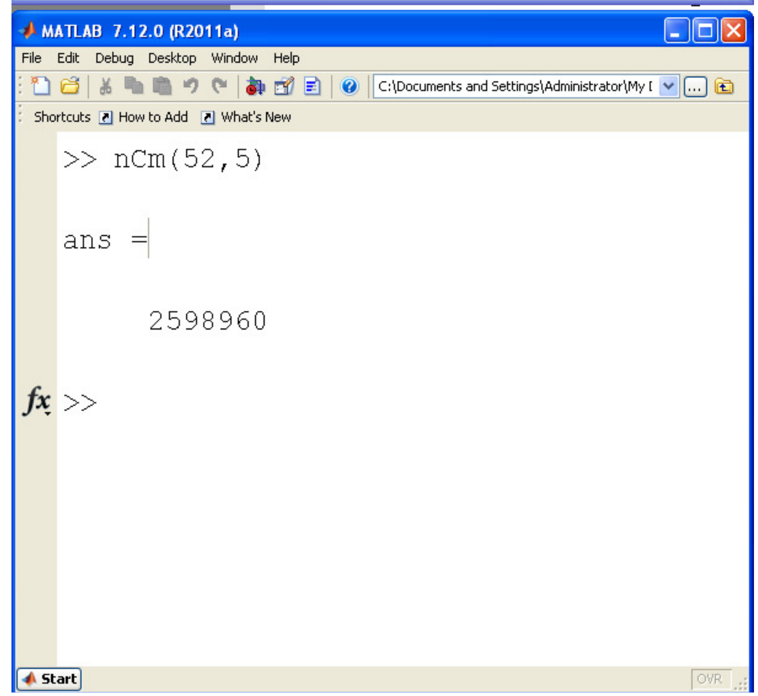
$$M = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$$

$$M = 5! = 120$$

The number of hands in poker is then

$$_{52}C_5 = \binom{52}{5} = \#hands = \left(\frac{52!}{(52-5)! \cdot 5!}\right) = 2,598,960$$





Probability of a Royal Flush

- order doesn't matter

The probability of any given hand as

$$p(x) = \frac{\text{the total number of hands that are x}}{2{,}598{,}960}$$

If you can compute the number of ways of getting a hand, you know the probability

From enumeration, N = 4

- Four ways to get a royal flush
- How to compute?

Probability of a Royal Flush

Use enumeration

- There are only four possibilities
  - Spades, hearts, clubs, and diamonds
- $p(\text{royal flush}) = \left(\frac{4}{2{,}598{,}960}\right) = \frac{1}{649{,}740}$

The odds against getting a royal flush are 649,740 : 1

You should on average get a royal flush once every 649,740 hands.

4 of a kind:

From enumeration, there are 624 ways to get a 4-of-a-kind

- How to compute?

4 of a kind (cont'd)

There are several ways to compute this:

Assume the cards in your hand are x-x-x-x-y

- There are 13 cards in a deck: choose one
- Of the four cards of that value, choose four
- Of the 48 remaining cards, choose one

$$N = \binom{13}{1}\binom{4}{4}\binom{48}{1} = 624$$

There are 624 different hands that give you a 4-of-a-kind.

- The odds being dealt 4 of a kind is 4165 : 1

$$p(N) = \left(\frac{624}{2{,}598{,}960}\right) = \left(\frac{1}{4165}\right)$$

3-of-a-kind

- Enumeration: N = 54,912

3-of-a-kind: xxx a b (N = 54,912)

- Of the 13 value, choose one for x
- Of the four cards of that type, choose three
- Of the 12 remaining values, choose two for a & b
  - When elements have the same frequency, do together
- Of the four a's, choose 1
- Of the four b's, choose 1

$$N = \binom{13}{1}\binom{4}{3}\binom{12}{2}\binom{4}{1}\binom{4}{1} = 54,912$$

The odds are 47.33 : 1 against

$$p(N) = \left(\frac{54,912}{2,598,960}\right)$$

Flush

- Five cards of one suit
- From Wikipedia, N = 5108

Flush (N = 5108)

- Of the four suits, choose one
- Of the 13 cards in that suit, choose five

$$N = \binom{4}{1}\binom{13}{5} = 5,148$$

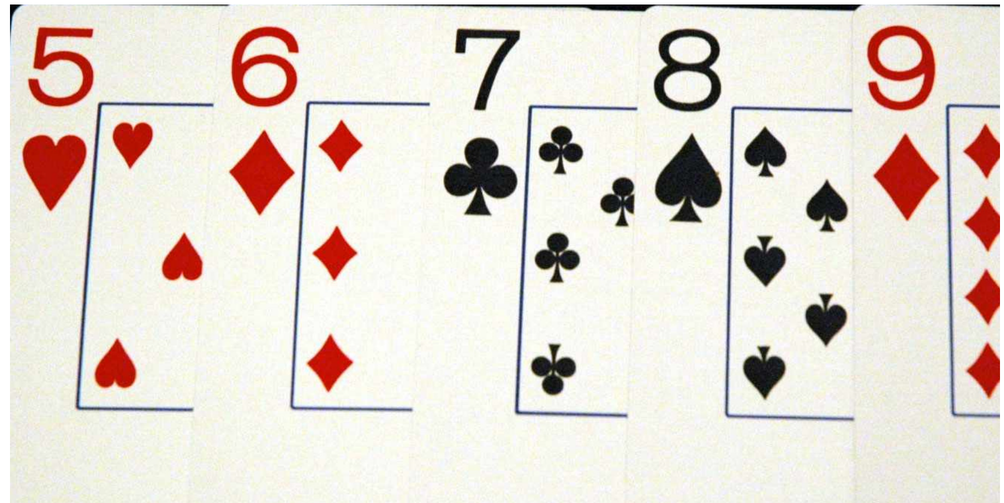
- This includes straight-flushes (N = 40).
  - Removing these gives 5,108 different flushes

The odds 504.8 : 1 against

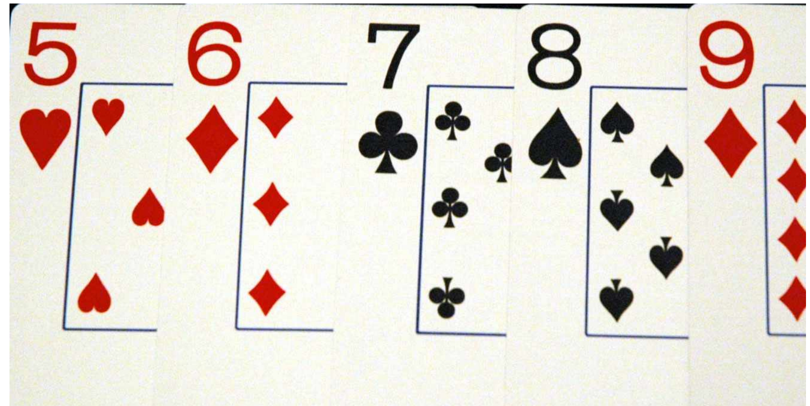
$$p(N) = \left(\frac{5,108}{2,598,960}\right) = \left(\frac{1}{504.8}\right)$$

Straight

- A run of five cards
- 2 or more different suits
- Enumeration: N = 10,200



Straight (N = 10,200)



- A straight can start with an Ace through a 10.
  - That gives 40 starting cards (10 values in 4 suits)
- Of the four cards of the next value, choose one
- Of the four cards of the next value, choose one

so

$$N = 40 \cdot \binom{4}{1}\binom{4}{1}\binom{4}{1}\binom{4}{1} = 10,240$$

This also counts straight-flushes (40), so the total number of straights is

$$N = 10,200$$

giving the probability of 254.8:1 against

2-Pair

- Hand = xx yy z
- N = 123,552 (enumeration)

2-Pair (N = 123,552)

- Of the 13 values (Ace through King), pick two
- For the first value, pick two cards of the four in the deck
- For the second value, pick two cards of the four in the deck
- For the last card, pick one (44 choose 1)

$$N = \binom{13}{2}\binom{4}{2}\binom{4}{2}\binom{44}{1}$$

$$N = 123,552$$

The odds of getting 2-pair are

$$p(x) = \left(\frac{123{,}552}{2{,}598{,}960}\right) = \left(\frac{1}{21.035}\right)$$

or 21.03 : 1 odds against getting two-pair.

Pair (N = 1,098,240)

- xx a b c

N = (13 choose 1 for x) * (4 x's choose 2)

* (12 choose 3 for a/b/c) * (4a's choose 1) * (4b's choose 1)*(4c's choose1)

$$N = \binom{13}{1}\binom{4}{2} \cdot \binom{12}{3}\binom{4}{1}\binom{4}{1}\binom{4}{1}$$

$$N = 1,098,240$$

The odds of being dealt a pair is 2.366 : 1 against

$$p(x) = \left(\frac{1,098,240}{2,598,960}\right) = \left(\frac{1}{2.366}\right)$$

Probability of all poker hands

https://allmathconsidered.wordpress.com/2017/05/23/the-probabilities-of-poker-hands/

POKER HAND		COUNT	Odds Against
2	Straight Flush	40	64,974
3	Four of a Kind	624	4,165
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Total		2,598,960	

Challenge: What are the odds if you play with two decks

- Two 52-card decks
- 104 cards total

How many different poker hands are there?

- Deal 5 cards
- Order doesn't matter

What are the odds of 4-of-a-kind?

- 104-card deck

4-of-a-Kind with Two Decks

Hands

$$M = \binom{104}{5} = 91{,}962{,}520$$

4-of-a-kind (xxxx y)

$$N = \binom{13}{1}\binom{8}{4} \cdot \binom{12}{1}\binom{8}{1}$$

$$N = 87{,}360$$

In 100,000 hands, you should get 94.99 4-of-a-kind hands

$$n = \left(\frac{87{,}360}{91{,}962{,}520}\right) 100,000 = 94.99$$

What are the odds of a full-house?

- 104-card deck

What are the odds of a full-house?

Hands

$$M = \binom{104}{5} = 91{,}962{,}520$$

Full-House (xxx yy)

$$N = \binom{13}{1}\binom{8}{3} \cdot \binom{12}{1}\binom{8}{2}$$

$$N = 244,608$$

In 100,000 hands, you expect 266 full-houses

$$n = \left(\frac{244{,}608}{91{,}962{,}520}\right) \cdot 100,000$$

$$n = 265.98$$

How to check your answer?

Enumeration takes too long

- 30 minutes for a 52-card deck
- 16 hours for a 104-card deck

Monte-Carlo will get you close

```
Pair4 = 0;
FullHouse = 0;
Pair3 = 0;
Pair22 = 0;
Pair2 = 0;

for i0 = 1:1000

X = rand(1,52);
[a,Deck] = sort(X);
Hand = Deck(1:5);
Value = mod(Hand,13) + 1;
Suit = floor(Hand/13) + 1;

N = zeros(1,13);
for n=1:13
    N(n) = sum(Value == n);
end

[N,a] = sort(N);

if (N(13) == 4) Pair4 = Pair4 + 1; end
if ((N(13) == 3)*(N(12) == 2)) FullHouse=FullHouse +
if ((N(13) == 3)*(N(12) < 2)) Pair3 = Pair3 + 1; end
if ((N(13) == 2)*(N(12) == 2)) Pair22 = Pair22 + 1;
if ((N(13) == 2)*(N(12) < 2)) Pair2 = Pair2 + 1; end

end
```

Monte-Carlo Simulations

Monte-Carlo code to play poker with two decks

- A 104-card deck is shuffled
- A 5-card hand is dealt
- The type of hand it is then logged
- This experiment is then repeated a large number of times

The odds are then approximately the number of times each hand occurred divided by the number of hands dealt.

Monte-Carlo Procedure:

Shuffle two decks of cards (52 cards in each deck):

```
X = rand(1,52*2);
[a,Deck] = sort(X);
Deck = Deck - 1;
Deck(1:10)
     18    69    70    47    65    41    14    85     7    22
```

Take each card mod 52

```
Deck = mod(Deck, 52);
Deck(1:10)
     18    17    18    47    13    41    14    33     7    22
```

Draw a 5-card hand

```
>> Hand = Deck(1:5)

Hand =    18    17    18    47    13
```

Determine the value and suit of each card:

```
>> Value = mod(Hand, 13) + 1

Value =     6     5     6     9     1

>> Suit = floor(Hand/13) + 1

Suit =     2     2     2     4     2
```

Your hand is

- 6 of diamonds
- 5 of diamonds
- 6 of diamonds
- 9 of spades
- Ace of diamonds

For pairs, count the frequency of each card

```
N = zeros(1,13);
for n=1:13
    N(n) = sum(Value == n);
end
```

Then determine what kind of hand by

- Sorting the number of matching cards
- Checking if the maximum of the matches is 4 (4-of-a-kind), 3 and 2 (full house), etc

```
[N,a] = sort(N, 'descend');

if (N(1) == 4) Pair4 = Pair4 + 1; end
if ((N(1) == 3)*(N(2) == 2)) FullHouse=FullHouse + 1; end
if ((N(1) == 3)*(N(2) < 2)) Pair3 = Pair3 + 1; end
if ((N(1) == 2)*(N(2) == 2)) Pair22 = Pair22 + 1; end
if ((N(1) == 2)*(N(2) < 2)) Pair2 = Pair2 + 1; end
```

Matlab Code

Initialize counters

repeat 100,000 times

shuffle two decks

deal out 5 cards

determine values & suits

count frequency of each card

check of each type of hand

```
Pair4 = 0;
FullHouse = 0;
Pair3 = 0;
Pair22 = 0;
Pair2 = 0;

tic
for i0 = 1:1e5

    X = rand(1,52*2);
    [a,Deck] = sort(X);
    Deck = Deck - 1;
    Deck = mod(Deck, 52);
    Hand = Deck(1:5);
    Value = mod(Hand,13) + 1;
    Suit = floor(Hand/13) + 1;

    N = zeros(1,13);
    for n=1:13
        N(n) = sum(Value == n);
    end

    [N,a] = sort(N, 'descend');

    if (N(1) == 4) Pair4 = Pair4 + 1; end
    if ((N(1) == 3)*(N(2) == 2)) FullHouse=FullHouse + 1;
    if ((N(1) == 3)*(N(2) < 2)) Pair3 = Pair3 + 1; end
    if ((N(1) == 2)*(N(2) == 2)) Pair22 = Pair22 + 1; enc
    if ((N(1) == 2)*(N(2) < 2)) Pair2 = Pair2 + 1; end

end
toc
disp('4 of a kind, Full House, 3 of a kind, 2 Pair, Pair'
disp([Pair4, FullHouse,Pair3,Pair22,Pair2])
```

Monte-Carlo Results

- One Deck (known odds)
- Always test your code

The results for 1 million hands of poker:

```
              4-of-kind  full-house   3-o-k         2-pair      pair
  X =           252        1409       21097        47498      423055
```

The odds match up (per 1,000,000 hands)

Hand	Computed	Simulated
4 of a kind	240.1	252
Full-House	1,440.9	1,409
3 of a kind	21,141	21,097
2-pair	47,619	47,489
pair	421,940	423,055

Monte-Carlo Results

- Monte-Carlo results match up with calculations
- Playing with two decks affects the odds

	One Deck 100,000 hands	Two Decks 100,000 hands	
Hand	Calculations	Calculations	Simulated
4 of a kind	24	94.99	101
Full-House	144.1	265.98	295
3 of a kind	2,114.1	?	3,357
2-pair	4,761.9	?	5,752
pair	42,194	?	44,485

Summary

With combinatorics, you can calculate how many combinations result in a given outcome

- 4 of a kind
- full house
- 3 of a kind

Where there are a large number of possibilities, this cans save a lot of time

- Enumeration can take hours or days to go through every permutation