Enumeration

ECE 341 Random Processes

Lecture #3

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Enumeration

Probability is defined as the number of times an event occurs as the number of trials goes to infinity.

This leads to the previous lecture

- Monte Carlo experiments

A second method is enumeration

- Assume all outcomes have equal probability
- The *exact* probability of an event is then

$$p(x) = \left(\frac{\text{number of ways to obtain outcome x}}{\text{total number of possible outcomes}} \right)$$

Enumeration is a brute-force technique.

- In some cases, it works very well.
- In other cases, there are simply too many possible outcomes

Topics in This Lecture

Solve the same problems we solved using Monte-Carlo techniques:

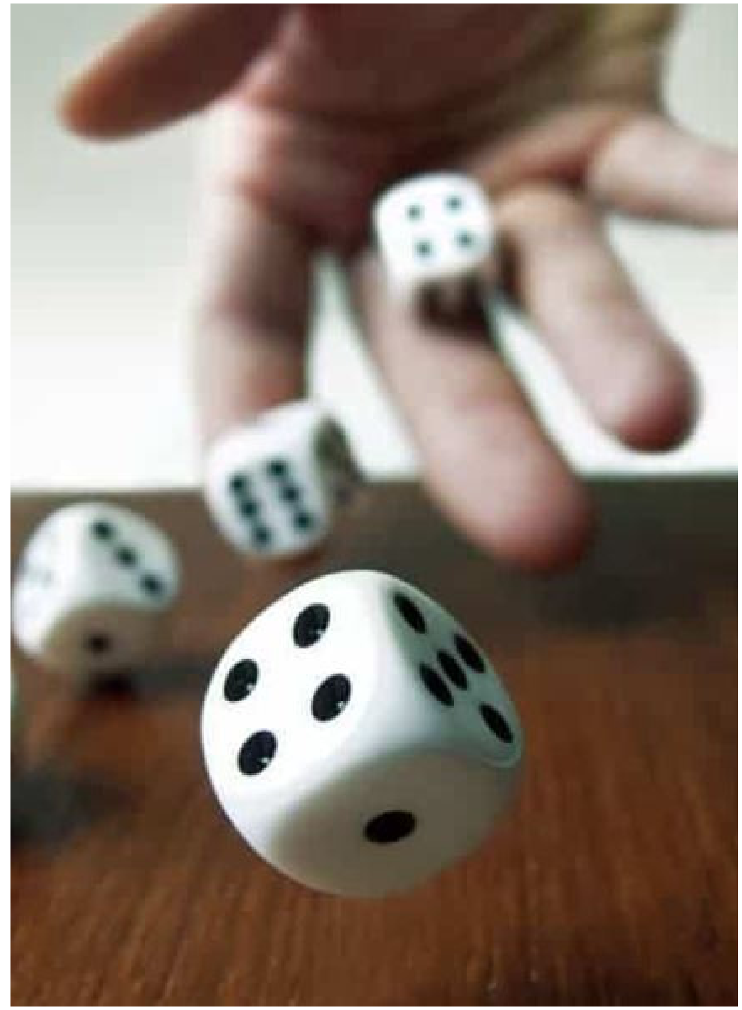
- Probability of rolling a 1 on a 6-sided die
- Probability distribution of the max(d4, d6)
- Max of (d4, d6) vs. d6
- 5-Game Match (tree diagram),
- Rolling 6-Dice
- Drawing a full-house or 3-of-a-kind in poker

Case 1: Rolling a single 6-sided die (d6)

What's the probability of rolling a 1 on a 6-sided die?

Previous lecture with 1 million rolls

- Trial 1: 166,219 times
- Trial 2: 167,090 times
- Trial 3: 166,969 times



From these results,

$$p \approx 0.166 \quad \text{(ish)}$$

That's one problem with Monte Carlo simulations

- Results are approximate
- You *can* place a bound on the actual probability
  - Student-t test
  - Future topic

Case 1: Enumeration

Assume all outcomes have equal probability.

List all possible outcomes

- {1, 2, 3, 5, 6}
- N = 6

Count how many are successes

- {1}
- M = 1

Hence, the probability of rolling a one is

$$p = \frac{M}{N} = 1/6$$

Note

- This matches up with Monte Carlo experiments
- The answer is exact

Case 2: A = max(d4, d6)

Player A rolls two dice

- { d4, d6 }

A's score is the maximum of the two

What is the probability A scores 1..6 points?

Case 2 (cont'd)

Step 1: List out all possibilities

- There are 24 possibilities (N = 24)

(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1.6)
(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)

Step 2: List all possible ways to get each outcome:

- 1: (1,1)
- 2: (2,1), (2,2), (2,3)
- 3: (3,1), (3,2), (3,3), (2,3), (1,3)
- 4: (1,4), (2,4), (3,4), (4,4), (4,3), (4,2), (4,1)
- 5: (1,5), (2,5), (3,5), (4,5)
- 6: (1,6), (2,6), (3,6), (4,6)

Case 2: (cont'd)

The odds are then

- The number of ways to get each outcome
- Divided by the total number of possible outcomes

x (Score)	Number of Ways to Get this Score	p(x)
1	1	1/24
2	3	3/24
3	5	5/24
4	7	7/24
5	4	4/24
6	4	4/24

p(x) is termed *the probability density function*

Case 3: max(d4, d6) vs. d6 (conditional probability)

A third example is this:

- A rolls a d4 and a d6
  - Takes the higher score
- B rolls a d6

Highest score wins

- B wins on ties.

What is the probability that A will win this game?

- Monte Carlo: $p \approx 0.486$

Case 3: (cont'd)

Problem: Too many possible outcomes

- N = 4 * 6 * 6 = 144

Solution: Use conditional probabilities

- Divide-and-conquer technique
- Split this into six smaller problems

p(A) =

- p(A|B=1) p(B=1) +
- p(A|B=2) p(B=2) +
- p(A|B=3) p(B=3) +
- p(A|B=4) p(B=4) +
- p(A|B=5) p(B=5) +
- p(A|B=6) p(B=6)

Case 3: (cont'd)

B = 1:

- A has to score 2 or higher to win.
- There are 23 ways for A to score 2 or more points, meaning

$p(A|B=1) = \frac{23}{24}$ $\quad\quad$ $p(B=1) = \frac{1}{6}$

B = 2:

- A has to score 3 or higher to win.
- There are 20 ways for A to score 3 or more points, meaning

$p(A|B=2) = \frac{20}{24}$ $\quad\quad$ $p(B=2) = \frac{1}{6}$

B = 3:

- A has to score 4 or higher to win.
- There are 15 ways for A to score 4 or more points, meaning

$p(A|B=3) = \frac{15}{24}$ $\quad\quad$ $p(B=3) = \frac{1}{6}$

Case 3 (cont'd)

B = 4:

- A has to score 5 or higher to win.
- There are 8 ways for A to score 5 or more points, meaning

$p(A|B=4) = \frac{8}{24}$ $\qquad$ $p(B=4) = \frac{1}{6}$

B = 5:

- A has to score 6 or higher to win.
- There are 4 ways for A to score 6 or more points, meaning

$p(A|B=5) = \frac{4}{24}$ $\qquad$ $p(B=5) = \frac{1}{6}$

B = 6: A loses

$p(A|B=6) = 0$ $\qquad$ $p(B=6) = \frac{1}{6}$

Case 3 (cont'd)

Therefore, the probability that A wins is

$$p(A) = \left(\frac{23}{24}\right)\left(\frac{1}{6}\right) + \left(\frac{20}{24}\right)\left(\frac{1}{6}\right) + \left(\frac{15}{24}\right)\left(\frac{1}{6}\right) + \left(\frac{8}{24}\right)\left(\frac{1}{6}\right) + \left(\frac{4}{24}\right)\left(\frac{1}{6}\right) + \left(\frac{0}{24}\right)\left(\frac{1}{6}\right)$$

$$p(A) = \left(\frac{70}{144}\right) = 0.486111$$

This matches up with the Monte-Carlo simulations

- except that this answer is exact.

Case 4: 5-Game Match (Tree Analysis)

- Similar to Baseball or NBA finals

A and B are playing a match

- A has a 60% chance of winning any given game.
- Match consists of 5 games
- Whoever wins the most games wins the match

What is the chance that A wins the match?

Monte Carlo results:

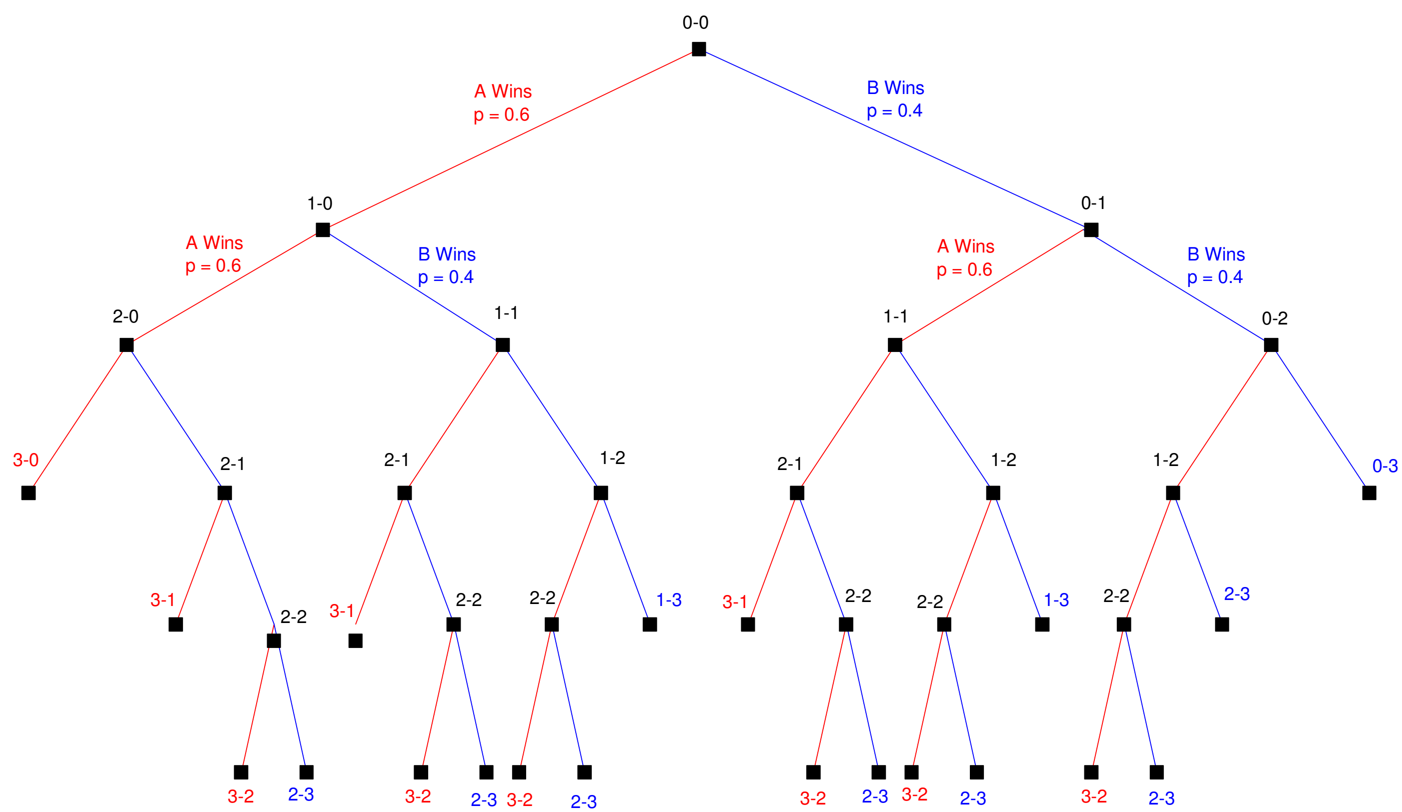
- p = 0.682 (ish)

Case 4: Tree Diagram

Each game has two possible outcomes:

- A wins (p = 0.6)
- B wins (p = 0.4)

List out all ways the series can proceed:



Case 4 (cont'd)

Count the number of ways A wins:

- 1 outcome ends 3-0
- 3 outcomes end in 3-1
- 6 outcomes end in 3-2

The odds of A winning the match are

$$p(A) = 1 \cdot p^3 + 3 \cdot p^3 q + 6 \cdot p^3 q^2$$

$$p(A) = 0.68256$$

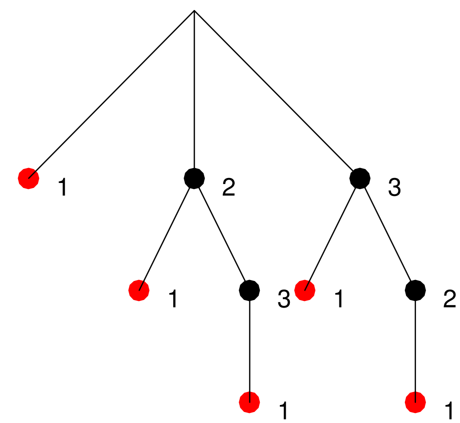
Note

- This matches up with Monte Carlo
- The answer is *exact*

Sidelight: Sampling With and Without Replacement

Tree diagrams for finite series

- First to win 3 games
- In a bin of 3 marbles (2 black, one red)
  - Pick one marble
  - Stop if it's red
  - If it's not red, leave it our and repeat

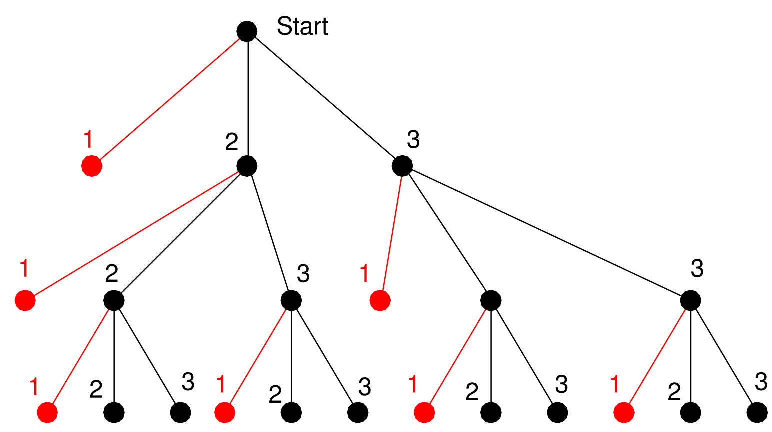


Tree diagrams do not work for infinite series

- First to win *by* 3 games
- In a bin of 3 marbles (2 black, one red)
  - Pick one marble
  - Stop if it's red
  - If it's not red, replace the marble
  - Repeat

For the latter, we need a different tool

- Markov chains
- Future topic



Enumeration with Matlab

Enumeration is a brute-force solution

- Go through every possible outcome
- Count how many of them were successes

With Matlab, you can write programs to grind out all possibilities using nested for-loops.

Case 5: Farkle (6d6)

Suppose you roll six 6-sided dice (6d6)

What are the odds of rolling

- Two triples (xxx yyy)?
- One triple (xxx aab or xxx abc)?

From Monte-Carlo

- Two triples: 6337 in 1,000,000 rolls
- One triple: 308,026 in 1,000,000 rolls

Case 5: Number of Rolls

There are 46,656 ways to roll 6d6.

- The first die has six possibilities
- The second die also has six possibilities
- etc

$$N = 6^6 = 46,656$$

That seems like a large number, but it's no problem for Matlab.

Case 5 Nested For-Loops

Start by going through every possible outcome

- Nested for-loops
- 46,656 different outcomes

```
for d1 = 1:6
   for d2 = 1:6
      for d3 = 1:6
         for d4 = 1:6
            for d5 = 1:6
               for d6 = 1:6
                  Roll = [d1, d2, d3, d4, d5, d6];
               end
            end
         end
      end
   end
end
```

Case 5: Determine the frequency

Once you roll the dice

- Find the frequency of each number
- Sort in decreasing order

```
Roll = [d1, d2, d3, d4, d5, d6];
F = zeros(1,6);
for i=1:6
   F(i) = sum(Roll == i);
end
F = sort(F, 'descend')

[Roll]
[F]
```

script window

Example:

- Roll = {2, 5, 2, 2, 5, 6}
- F(1) = 3
  - There are three 2's
- F(2) = 2
  - There are two 5's
  - Next highest frequency
- F(3) = 1
  - There is one 6

```
Roll = 2 5 2 2 5 6

F = 3 1 1 0 0 0
```

command window

Case 5: Determine the roll type

Once you know F(), you can determine the type of hand

```
if( (F(1) == 3)*(F(2)==3))
    Pair33 = Pair33 + 1;
    end
if( (F(1) == 3)*(F(2)<3))
    Pair3 = Pair3 + 1;
    end
```

By counting, you'll know the total number of hands that result in two and one triples.

Net Code

- Nested for-loops
- Goes through all combinations

```
Pair33 = 0;
Pair3 = 0;
N = 0;
for d1 = 1:6
  for d2 = 1:6
    for d3 = 1:6
      for d4 = 1:6
        for d5 = 1:6
          for d6 = 1:6
             Dice = [d1,d2,d3,d4,d5,d6];
             N = N + 1;
             F = zeros(1,6);
             for i=1:6
               F(i) = sum(Dice == i);
             end
             F = sort(F, 'descend');
             if( (F(1) == 3)*(F(2) == 3))
               Pair33 = Pair33 + 1;
             end
             if( (F(1) == 3)*(F(2) < 3))
               Pair3 = Pair3 + 1;
             end
           end
         end
       end
     end
   end
end
```

Case 5: Results

The results are

- Pair33 = 300
- Pair3 = 14400
- N = 46656
- Elapsed time is 1.972013 seconds.

There are

- 300 ways to get two triples,
- 14,000 ways to get one triple, and
- 46,656 total number of ways to roll three dice.

This took 1.97 seconds

- 3.4GHz Windows computer
- Not a problem for Matlab

Case 6: Enumeration with Card Games

Finally, let's use enumeration in poker

- 52 card deck
- Deal out 5 cards

Create nested for-loops

- Avoid duplication of cards

Go through every possible hand

- 2,598,960 total

```
N = 0;
for c1 = 1:52
  for c2 = c1+1:52
    for c3 = c2+1:52
      for c4 = c3+1:52
        for c5 = c4+1:52
          Hand = [c1,c2,c3,c4,c5] - 1;
          N = N + 1
        end
      end
    end
  end
end
[N]
```

Script Window

```
N = 25989690
```

Command Window

Case 6: Determine hand

Hand is card number

- 0 to 51

Value = card value

- 1..13
- Ace through King

Suit = Card suit

- 1..4
- Club, Diamond, Heart, Spade

Example:

- Card #1 is the 2 of clubs
- Card #7 is the 8 of clubs
- Card #9 is the 10 of clubs
- Card #22 is the 10 of diamonds
- Card #47 is the 9 of spades

```
Hand = [c1,c2,c3,c4,c5] - 1;
Value = mod(Hand, 13) + 1
Suit = floor(Hand/13) + 1
```

Script Window

```
Hand  =  1   7   9  22  47
Value =  2   8  10  10   9
Suit  =  1   1   1   2   4
```

Command Window

Case 6: Determine hand type

Once you have your hand

- Determine the frequency of each card
  - variable F()
- Sort in descending order
  - F(1) is highest frequency of cards
- Check hand type
  - 3 + 2 = full house
  - 3 + 1 = three of a kind

```
F = zeros(1,5);
for i=1:13
    F(i) = sum(Value == i);
end
F = sort(F, 'descend');
if( (F(1) == 3)*(F(2) == 2) )
    FH = FH + 1;
elseif( (F(1) == 3)*(F(2) < 2) )
    Pair3 = Pair3 + 1; end
end
```

Case 6: Resulting Matlab Code

Every possible poker hand

- Loops 2,598,960 times
- Takes 186 seconds to run

```
Pair3 = 0;
FH = 0;
N = 0;
for c1=1:52
  for c2 = c1+1:52
    for c3 = c2+1:52
      for c4 = c3+1:52
        for c5 = c4+1:52
          N = N + 1;
          Hand = [c1,c2,c3,c4,c5] - 1;
          Value = mod(Hand,13) + 1;
          Suit = floor(Hand/13) + 1;
          F = zeros(1,13);
          for n=1:13
            F(n) = sum(Value == n);
          end
          F = sort(F, 'descend');
          if ((F(1) == 3)*(F(2) == 2))
             FH = FH + 1;
          elseif ((F(1) == 3)*(F(2) < 2))
             Pair3 = Pair3 + 1;
          end
        end
      end
    end
  end
end
```

Case 6: Results

Net Result:

- 2,598,960 poker hands
- 3744 full-houses
- 54,912 three-of-a-kind

Results match with Monte Carlo

- Monte-Carlo is approximate
- Enumeration is exact

Results match with Wikipedia

- Poker has been analyzed to death

$$p(fh) = \left(\frac{3744}{2{,}598{,}960} \right) = 0.0014406$$

$$p(3ok) = \left(\frac{54{,}912}{2{,}598{,}960} \right) = 0.0211285$$

```
N =           2598960
FH =             3744
Pair3 =         54912
Elapsed time is 186.303521 seconds.
```

Summary

While Monte-Carlo simulations give you approximate probabilities, enumeration gives you exact probabilities.

Enumeration is a brute-force approach:

- You go through list out every possible outcome.

Assuming each outcome has equal probability,

$$p = \left(\frac{\text{the number of successful outcomes}}{\text{the total number of outcomes}}\right)$$

Sometimes, enumeration works well

Sometimes, enumeration doesn't work

- There are too many possible outcomes

For the latter case, we need a different tool

- Combinatorics
- Next lecture