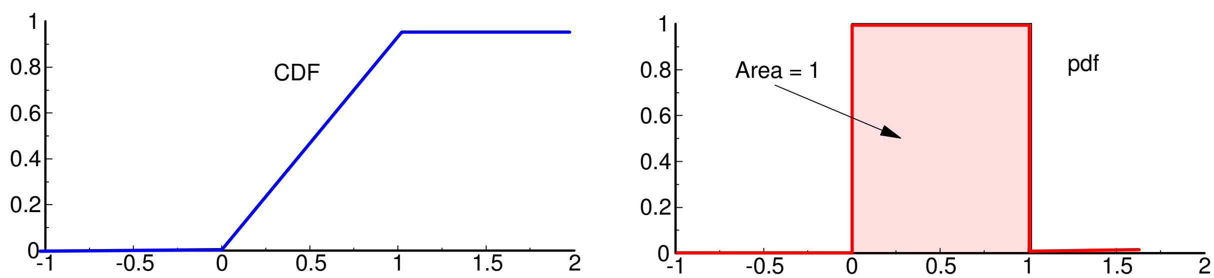
Uniform Distribution

Introduction:

A uniform distribution is one where the probability density function (pdf) is

- constant over a range (a, b), and
- zero otherwise.

For example, a variable X which has a uniform distribution function over the interval of (0,1) is as follows:



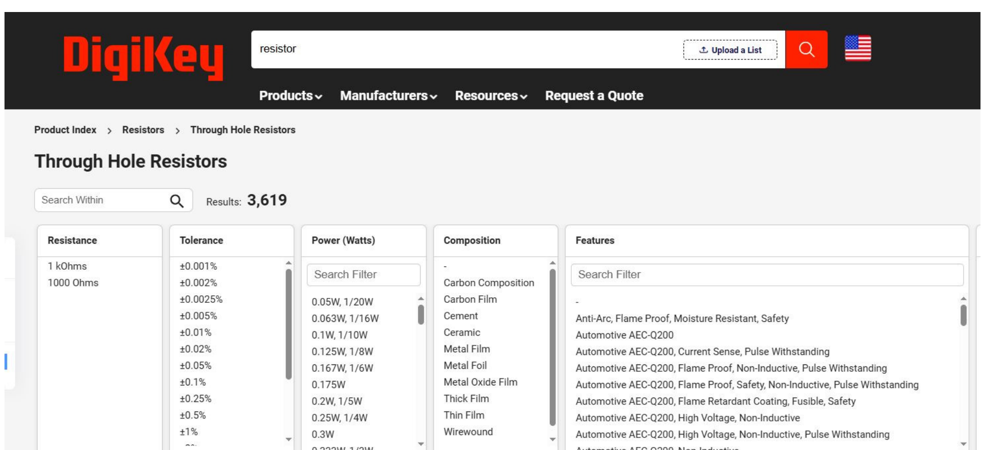
Uniform distribution over the range of (0,1)

There are several examples of where you encounter uniform distributions:

Matlab rand function: The *rand* funtion in Matlab returns a floating point number in the range if (0,1). This has the pdf as shown above

5% Tolerance Resistors: If you go to Digikey, you can find over 513,000 discrete through-hole resistors for sale. If you dive into the web site and select 1k resistors, you can find resistors with a tolerance of 0.001% up to +/- 20%. If you select 5% tolerance resistors, you still have 637 1k resistors to choose from.

2-%

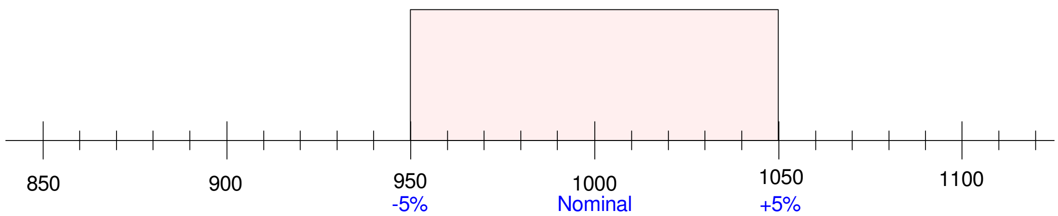


Options for 1k through-hole resistors from Digikey as of June 4, 2025

What the +/- 5% tolerance tells you is that any given resistor sold can be within the range of 1k, give or take 5% (50 Ohms). Likewise, all that's guaranteed is that any resistor from this lot will be in the range if (950, 1050) Ohms.

Given these specifications, there's some debate about how to model the variations of 5% resistors

- You could model this as a normal distribution (coming up) with tails truncated at +/- 5%
- You could model this as a uniform distribution over the range of 1k - 5% to 1k + 5% (shown below:



pdf for a 1k resistor with +/- 5% tolerance

There's also an argument that a better model would be a doughnut:

- The manufacturer may remove all resistors which are less than 2% away from nominal and sell these at a premium.
- This leaves a uniform distribution (above) with the section within 2% of nominal removed.

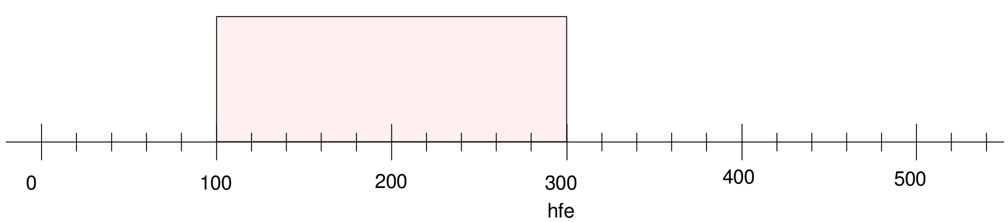
As far as which model is correct, this is something you can test using a chi-squared test (coming later in this course).

Transistor Gains: hfe **or** β: BJT transistors can be used as an electronic switch or as an amplifier. These circuits are covered in Electronics. In electronics, we typically use 2N3904TFR NPN transistors. If you look at the data sheets, you'll find that the current gain for Ic = 10mA is somewhere in the range of 100 to 300.

ON CHARACTERISTICS[(5)]				
h_{FE}	DC Current Gain	I_C = 0.1 mA, V_{CE} = 1.0 V	40	
		I_C = 1.0 mA, V_{CE} = 1.0 V	70	
		I_C = 10 mA, V_{CE} = 1.0 V	100	300
		I_C = 50 mA, V_{CE} = 1.0 V	60	
		I_C =100 mA, V_{CE} = 1.0V	30	

Current Gain for a 2N3904TFR NPN transistor

With nothing else to go on, you could assume that hFE has a uniform distribution over the range of (100,300).



Current gain for a 2N3904TFR transistor.

Whether or not these transistor actually *do* have a uniform distribution for hfe is something we can test when we get to chi-squared tests and F-tests (future lectures).

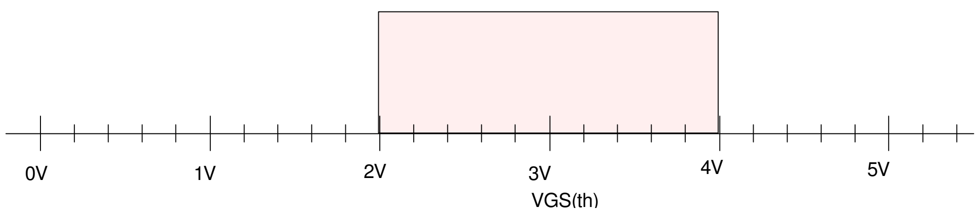
IRF530 N-Channel MOSFET: Like BJT transistors, N-channel Mosfet's can also be used as electronic switches or amplifiers. These too are covered in Electronics. One of the parameters that are important in Mosfet circuit design is the turn-on voltage. VGS(th). If you look at the data sheets for an IRF530 Mosfet, you'll find that only the min, typical, and max values of VGS(th) are given.

ON (*)

Symbol	Parameter	Test Conditions	Min.	Typ.	Max.	Unit
$V_{GS(th)}$	Gate Threshold Voltage	$V_{DS} = V_{GS}$ $I_D = 250\ \mu A$	2	3	4	V
$R_{DS(on)}$	Static Drain-source On Resistance	$V_{GS} = 10\ V$ $I_D = 7\ A$		0.115	0.16	Ω

Part of the data sheet for a IRF530 Mosfet

If you buy several IRF530 Mosfet's and measure VGS(th), each one will be slightly different. To model this, you could use a uniform distribution over the range of (2,4). Whether or not an actual shipment of these Mosfets have a uniform or some other distribution is some you could test using a chi-squared of F-test (future lectures).

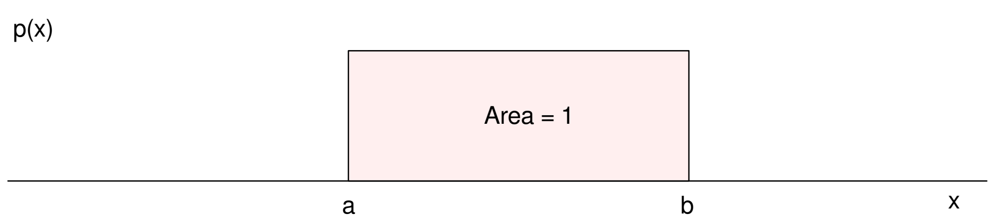


Variations in VGS(th) modeled as a uniform distribution

In general, if you know the limits but know nothing else about the distribution, a uniform distribution is a reasonable assumption to make about the random variable.

Properties of Uniform Distributions:

Assume a uniform distribution over the interval (a, b)



Since the area must be 1.000, the height must be

Area = width * height = 1

$$p(x) = \left(\frac{1}{b-a}\right) \qquad a < x < b$$

Mean: The mean of the function (almost by inspection) is

$$\mu = \left(\frac{a+b}{2}\right)$$

Variance: The variance is

$$\sigma^2 = \int_a^b p(x)(x-\bar{x})^2 dx$$

Do a change in variable (time-shift so that the pdf is symmetric about zero)

$$c = \frac{b-a}{2}$$

$$\sigma^2 = \int_{-c}^{c} p(x)(x-\bar{x})^2 dx$$

$$= \int_{-c}^{c} \left(\frac{1}{2c}\right)(x)^2 dx$$

$$= \left(\frac{1}{6c}\right) \left(x^3\right)_{-c}^{c}$$

$$= \left(\frac{c^2}{3}\right)$$

Substitute back for c

$$\sigma^2 = \left(\frac{(b-a)^2}{12}\right)$$

Moment Generating Function

$$\psi(s) = \left(\frac{1}{s}\right)\left(e^{-bs} - e^{-as}\right)$$

Combinations of Uniform Distributions

If you add two uniform distributions, the result is

- The convolution of their pdf's, or
- The product of their moment generating functions.

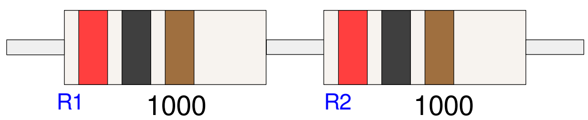
This could happen by

- Adding two uniform distributions in Matlab

```
A = rand;
B = rand;
Y = A+B;
```

- Placing two 1k 5% tolerance resistors in series.

```
R1 = 1000 * (1 + 0.05*(2*rand-1));
R2 = 1000 * (1 + 0.05*(2*rand-1));
Y = R1 + R2;
```



Convolution of two uniform distributions: Result of placing two resistors in series

Example 1: Assume A and B are uniform distributions over the interval (0, 1).

- Find the pdf of the sum Y = A + B.

There are several ways to solve:

- Convolution by hand: Y will be the convolution of the pdf for A and the pdf for B.
- Convolution using Matlab: Matlab can also convolute two continuous pdf's.
- Moment-Generating Functions: When you convolve pdf's you multiply their MGF's

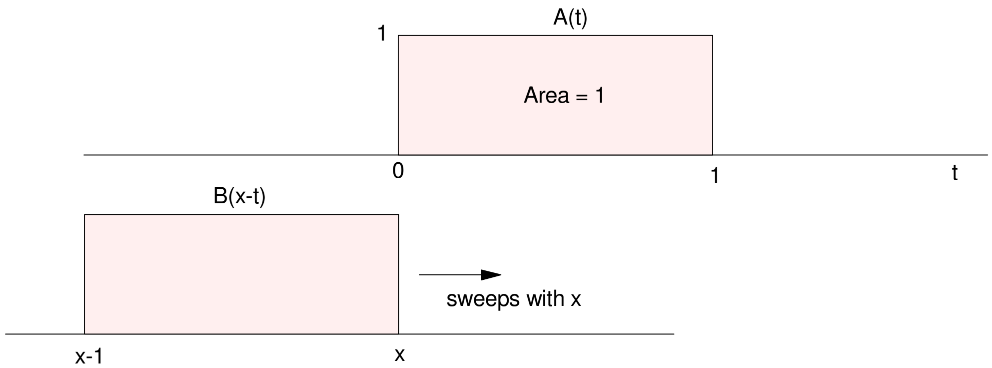
Using the first method, write the two pdf's and convolve them:

$$A(x) = u(x) - u(x-1)$$

$$B(x) = u(x) - u(x-1)$$

$$Y(x) = A(x) ** B(x)$$

$$Y(x) = \int_{-\infty}^{\infty} A(t)B(x-t)dt$$



Y is the convolution of two uniform distributions.

From the graph, this integral is

Y(x) =

- 0 x < 0
- x 0 < x < 1
- 1 - x 1 < x < 2
- 0 2 < x

This can be done with the functions as well

$$Y(x) = \int_{-\infty}^{\infty} A(t)\ B(x-t)\ dt$$

$$Y(x) = \int_{-\infty}^{\infty} (u(t-1) - u(t))\ (u(x-t-1) - u(x-t))\ dt$$

Note that B(x - t) is zero for

x - t < 0	t > x
x - t > 1	t < x-1

This allows us to simplify the integral

is one in the range of (0, 1), zero otherwise. This allows us to simplify:

$$Y(x) = \int_{x-1}^{x} (u(t) - u(t-1))\ dt$$

Integrating each part

$$Y(x) = (t\ u(t))_{x-1}^{x} - ((t-1)u(t-1))_{x-1}^{x}$$

solving should give your answer...

Solve using moment generating functions

$$A(s) = \left(\frac{1}{s}\right)(1 - e^{-s})$$

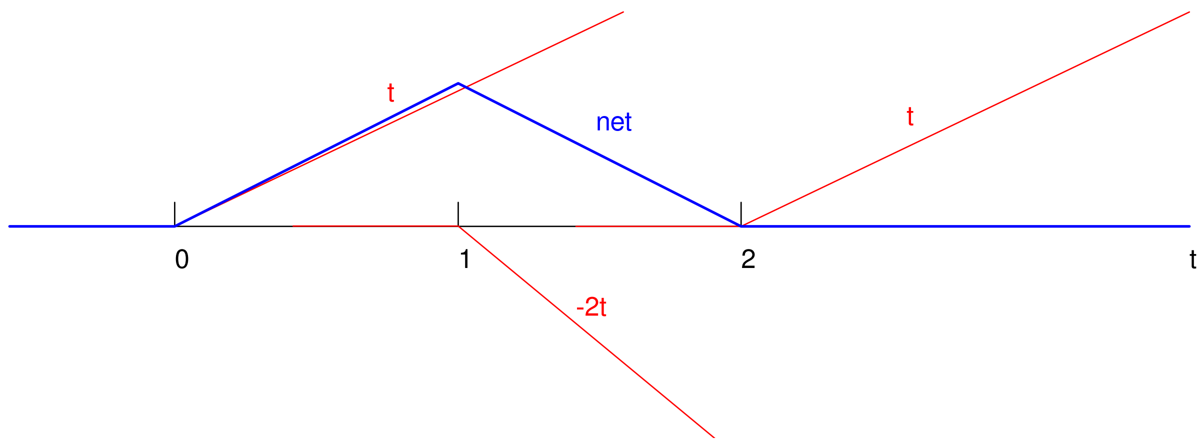
$$B(s) = \left(\frac{1}{s}\right)(1 - e^{-s})$$

$$Y(s) = A(s)B(s)$$

$$Y(s) = \left(\frac{1}{s^2}\right)(e^{-2s} - 2e^{-s} + 1)$$

Take the inverse LaPlace transform

$$y(x) = (x-2)\ u(x-2) - 2(x-1)\ u(x-1) + x\ u(x)$$



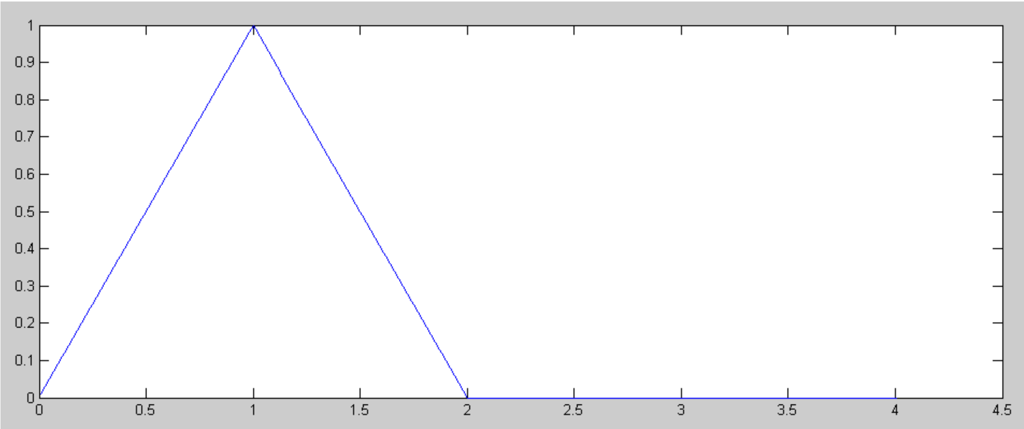
Solve using Matlab: Approximate a uniform distribution with 100 points over the interval (0, 1)

```
dx = 0.01;
x = [0:dx:2]';;

A = 1*(x<1);
B = 1*(x<1);

Y = conv(A, B) * dx;

plot([1:length(Y)]*dx,Y)
```



pdf for the sum of two uniform distributions

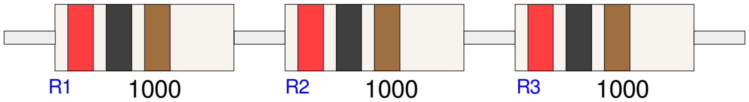
Combining Three Uniform Distributions

There are several ways to combine three uniform distributions. In Matlab, you could add three uniform distributions:

```
A = rand;
B = rand;
C = rand;
Y = A + B + C;
```

In lab, you could place three 1k 5% tolerance resistors in series.

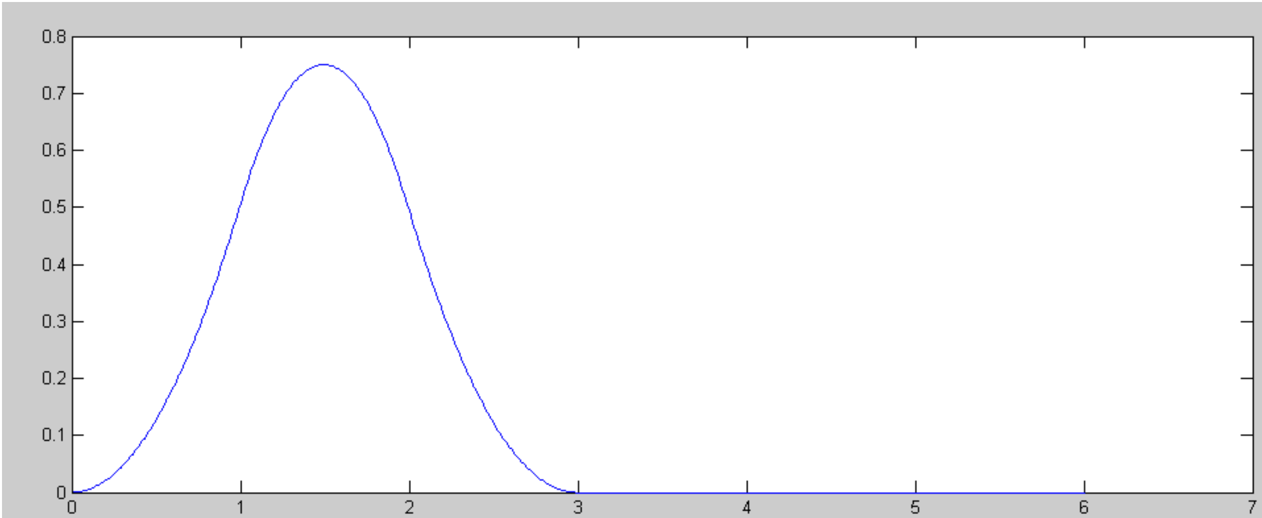
```
A = 1000 * (1 + 0.05*(2*rand-1));
B = 1000 * (1 + 0.05*(2*rand-1));
C = 1000 * (1 + 0.05*(2*rand-1));
Y = A + B + C;
```



Convolution of three uniform pdf's: the result of placing three 5% tolerance resistors in series.

In Matlab, can find the result using convolution:

```
x = [0:dx:2]';
A = 1*(x<1);
B = 1*(x<1);
C = 1*(x<1);
Y = conv(A, B) * dx;
Y = conv(Y, C) * dx;
plot([1:length(Y)]*dx,Y)
```



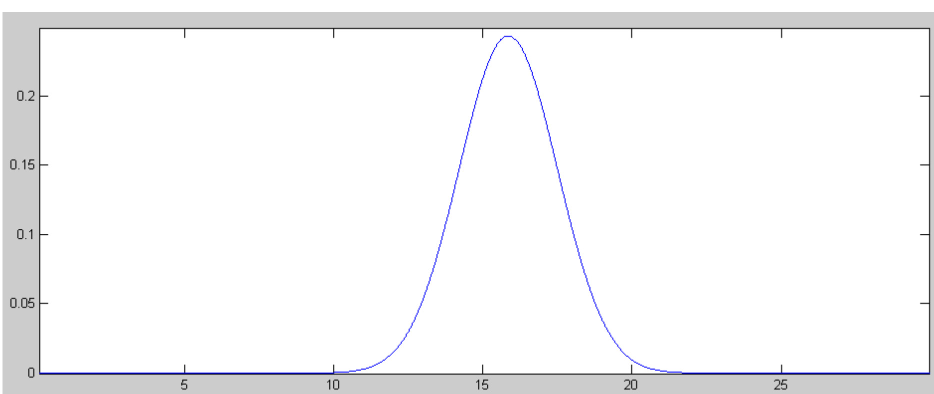
pdf for the sum of three uniform distributions

Note that after summing only three uniform distributions, you end up with something again has a bell-shaped curve. This again is due to the *Central Limit Theorem* and is a future coming soon...

Combining 32 uniform distributions:

If you sum 32 uniform pdf's (or place 32 resistors in series), the resulting pdf can be found using convolution:

```
x = [0:dx:2]';
A = 1*(x<1);
Y2 = conv(A, A) * dx;
Y4 = conv(Y2, Y2) * dx;
Y8 = conv(Y4, Y4) * dx;
Y16 = conv(Y8, Y8) * dx;
Y32 = conv(Y16, Y16) * dx;
plot([1:length(Y32)]*dx,Y32)
```

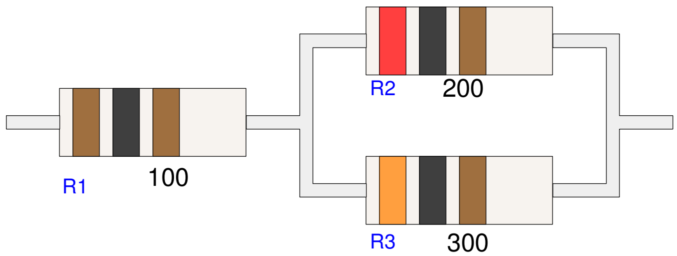


pdf for the sum of 32 uniform distributions

Note that the central limit theorem is evident here: the distribution is approaching a normal distribution.

Uniform Distribution in Circuit Analysis:

Suppose I have the following circuit where each resistor has a 5% tolerance. What is the pdf (or cdf) of the net resistance?



Find the pdf for the net resistance when each resistor has 5% tolerance

From Circuits I, the net resistance is

$$R = R_1 + R_2 || R_3$$

$$R = R_1 + \left(\frac{1}{R_2} + \frac{1}{R_3}\right)^{-1}$$

This creates a problem: when you add resistors, you convolve the pdf's. What happens when you multiply or divide?

The answer isn't clear. While you may not be able to find the net pdf through convolution, you can always find the result using a Monte Carlo simulation. Sorting the resulting net resistances results in the system's cdf.

Matlab Code:

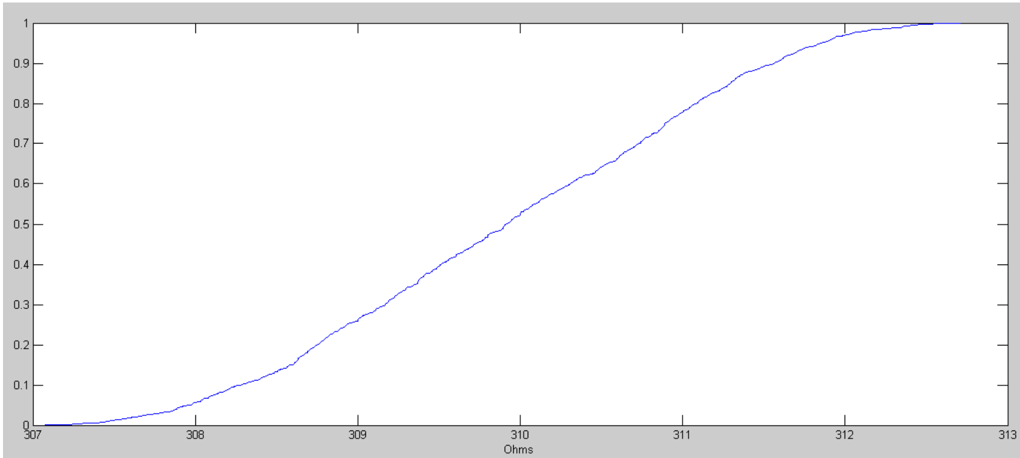
```
DATA = [];

for i=1:1000
   R1 = 190 * (1 + (rand()*2-1)*0.01);
   R2 = 200 * (1 + (rand()*2-1)*0.01);
   R3 = 300 * (1 + (rand()*2-1)*0.01);
   Rnet = R1 + 1 / (1/R2 + 1/R3);
   DATA = [DATA; Rnet];
end

DATA = sort(DATA);

p = [1:length(DATA)]' / length(DATA);
plot(DATA, p)
xlabel('Ohms')
```

The resulting cdf is as follows:



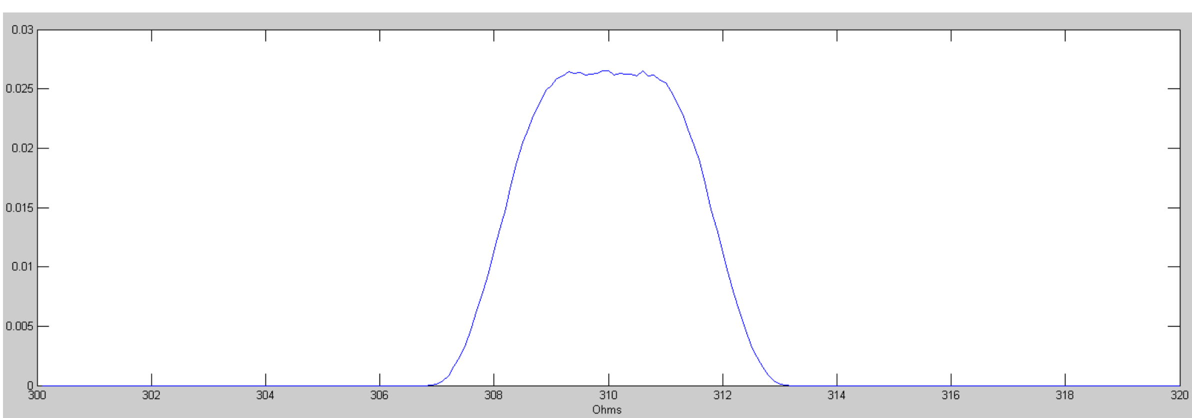
cdf for the net resistance

The pdf can be found by defining bins every 0.1 Ohm and counting how many times the data falls into each bin starting from 300 Ohms

```
X = zeros(200,1);

for i=1:1e6
   R1 = 190 * (1 + (rand()*2-1)*0.01);
   R2 = 200 * (1 + (rand()*2-1)*0.01);
   R3 = 300 * (1 + (rand()*2-1)*0.01);
   Rnet = R1 + 1 / (1/R2 + 1/R3);
   x = round((Rnet-300)*10);
   X(x) = X(x) + 1;
end
X = X / 1e6;

subplot(111)
plot([1:200]/10 + 300, X);
xlabel('Ohms')
```

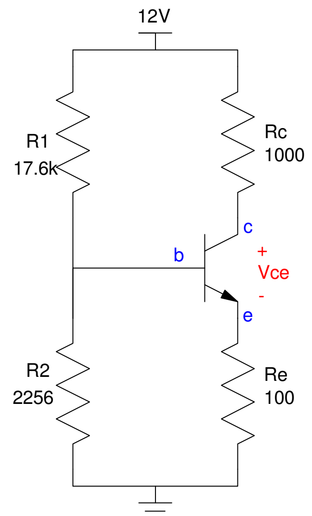


pdf for the net resistance

Example #3: BJT Amplifier. Consider next the following circuit from Electronics. Assume

- Resistors have a 5% tolerance
- The transistor has a gain in the range of 100 to 300

What will the variations in Vce be given these tolerances? What is the resulting cdf (or pdf)?



BJT Circuit from Electronics

The equations for this circuit (from Electronics)

$$V_{th} = \left(\frac{R_2}{R_2+R_1}\right)12V$$

$$R_{th} = \left(\frac{R_1 R_2}{R_1+R_2}\right)$$

$$I_b = \left(\frac{V_{th}-.07}{R_{th}+(1+\beta)R_e}\right)$$

$$I_c = \beta I_b$$

$$V_{ce} = 12 - R_c I_c - R_e(I_b + I_c)$$

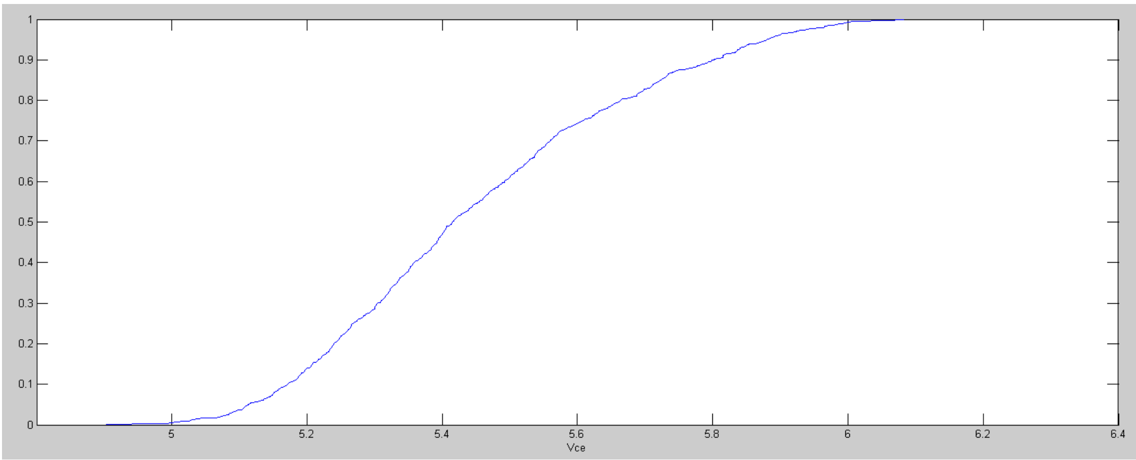
Using a Monte-Carlo simulation, you can find the cdf and the pdf.

The cdf comes from

```
DATA = [];

for i=1:1000
   R1 = 17600 * (1 + (rand()*2-1)*0.05);
   R2 = 2256 * (1 + (rand()*2-1)*0.05);
   Rc = 1000 * (1 + (rand()*2-1)*0.05);
   Re = 100 * (1 + (rand()*2-1)*0.05);
   Beta = 200 + 100*(rand()*2-1);
   Vb = 12*(R2 / (R1+R2));
   Rb = 1/(1/R1 + 1/R2);
   Ib = (Vb-0.7) / (Rb + (1+Beta)*Re);
   Ic = Beta*Ib;
   Vce = 12 - Rc*Ic - Re*(Ic+Ib);
   DATA = [DATA; Vce];
end

DATA = sort(DATA);
p = [1:length(DATA)]' / length(DATA);
plot(DATA, p)
```



cdf for the voltage Vce

The pdf comes from the following code:

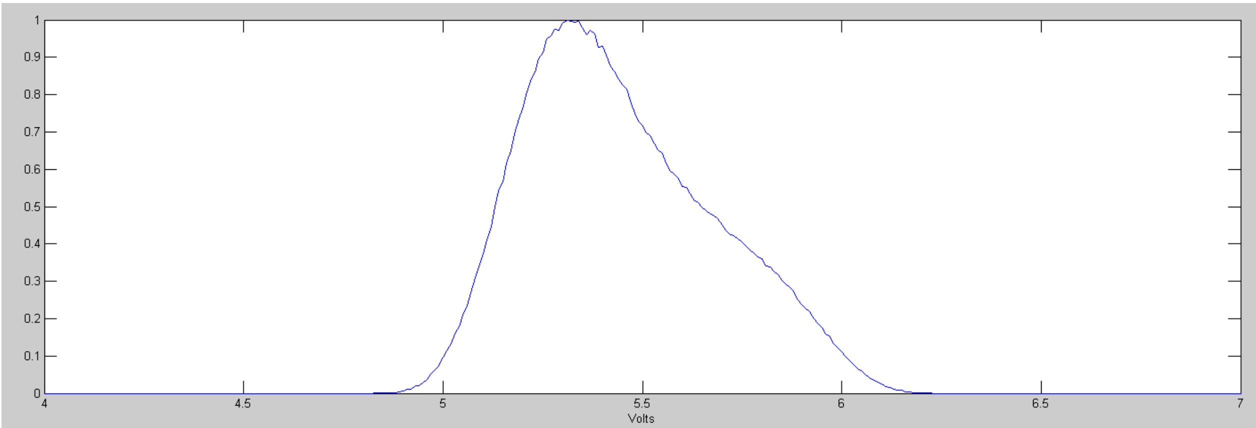
```
Data = zeros(1000,1);

for i=1:1e6
   R1 = 17600 * (1 + (rand()*2-1)*0.05);
   R2 = 2256 * (1 + (rand()*2-1)*0.05);
   Rc = 1000 * (1 + (rand()*2-1)*0.05);
   Re = 100 * (1 + (rand()*2-1)*0.05);
   Beta = 200 + 100*(rand()*2-1);
   Vb = 12*(R2 / (R1+R2));
   Rb = 1/(1/R1 + 1/R2);
   Ib = (Vb-0.7) / (Rb + (1+Beta)*Re);
   Ic = Beta*Ib;
   Vce = 12 - Rc*Ic - Re*(Ic+Ib);
   Bin = round(Vce*100);
   Data(Bin) = Data(Bin) + 1;
end

Data = Data / max(Data);

V = [1:1000]'/100;
plot(V, Data);
xlabel('Volts');

xlim([4,7])
```



pdf for the voltage Vce (scaling sets the max value to 1.00)

Summary

Uniform distributions have a constant probability over the range of (a, b) and can take on any value within that range. With a uniform distribution, you can model the effect of tolerances on electrical components such as resistors, transistors, Mosfets, etc. For some circuits, such as resistors in series, the resulting distribution can be found using convolution or moment generating functions. For more complex circuits, Monte Carlo techniques are almost required to find the resulting pdf.