Continuous Probability Density Functions

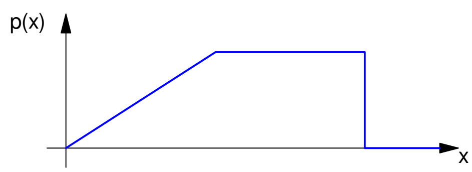
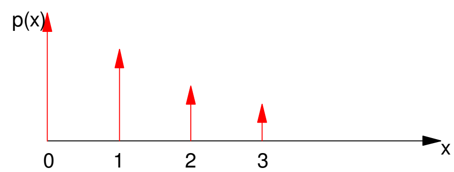
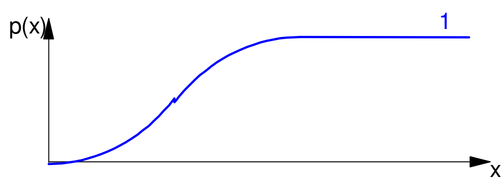
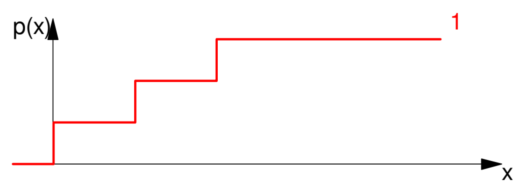
Up to now, we've been looking at discrete probability density functions. For these, the outcomes can only assume discrete values, typically integers, such as the number of times you roll a die until you get a 1.

From this point onward, we'll be looking at continuous probability density functions. Examples of these include

- The value of a 1k resistor
- The gain of a 3904 transistor
- The resistance of a thermistor at 25C
- The hottest temperature in May,
- etc.

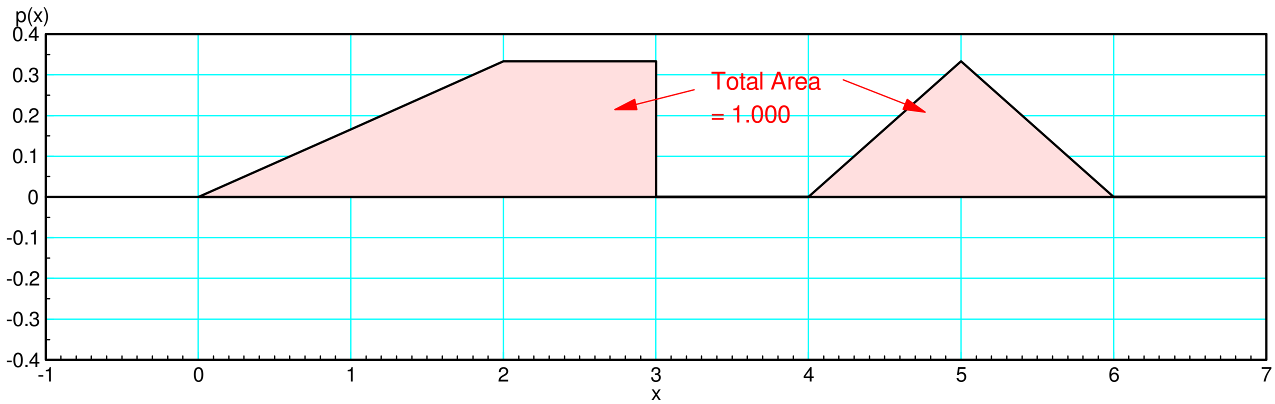
For each of these cases, the variable can assume any value over a range.

Most of what we did with discrete-time probabilities apply to continuous time - with just a slight change. Typically, this change is using LaPlace transforms rather than z-transforms for the moment generating function. A summary of these differences is

	Continuous	Discrete
pdf probability density function pdf = derivative of cdf		
cdf cumulative density function cdf = integral of pdf		
mgf moment generating function	$mgf = \psi(s)$	$mgf = \psi(z)$
m0 zeroth moment Area = 1	$m_0 = \psi(s=0) = 1$	$m_0 = \psi(z=1) = 1$
m1 1st moment mean	$m_1 = \psi'(s=0)$	$m_1 = \psi'(z=1)$
m2 2nd moment	$m_2 = \psi''(s=0)$	$m_2 = \psi''(z=1)$
variance	$\sigma^2 = m_2 - m_1^2$	$\sigma^2 = m_2 - m_1^2$

Example of a Continuous pdf

To illustrate what a continuous probability density function looks like, consider the following pdf:



Example pdf: p(x)

For this to be a valid pdf, the total area must be 1.000

$$\int_{-\infty}^{\infty} p(x)\, dx = 1$$

making the maximum value 0.3333.

Implementation in Matlab

To implement this pdf, it's actually easier to use the cdf (the integral of the pdf)

$$c(x) = \int_{-\infty}^{x} p(y)\, dy$$

Expressing p(x) mathematically:

$$p(x) = \begin{cases} 0 & x < 0 \\ x/6 & 0 < x < 2 \\ 1/3 & 2 < x < 3 \\ 0 & 3 < x < 4 \\ (x-4)/3 & 4 < x < 5 \\ (6-x)/3 & 5 < x < 6 \\ 0 & 6 < x \end{cases}$$

The cdf is then the integral of this (note that the integration constant maintain continuity)

$$cdf(x) = \begin{cases} 0 & x < 0 \\ \frac{x^2}{12} & 0 < x < 2 \\ \frac{x-2}{3} + \frac{1}{3} & 2 < x < 3 \\ \frac{2}{3} & 3 < x < 4 \\ \frac{(x-4)^2}{6} + \frac{2}{3} & 4 < x < 5 \\ 1 - \frac{(6-x)^2}{6} & 5 < x < 6 \\ 1 & 6 < x \end{cases}$$

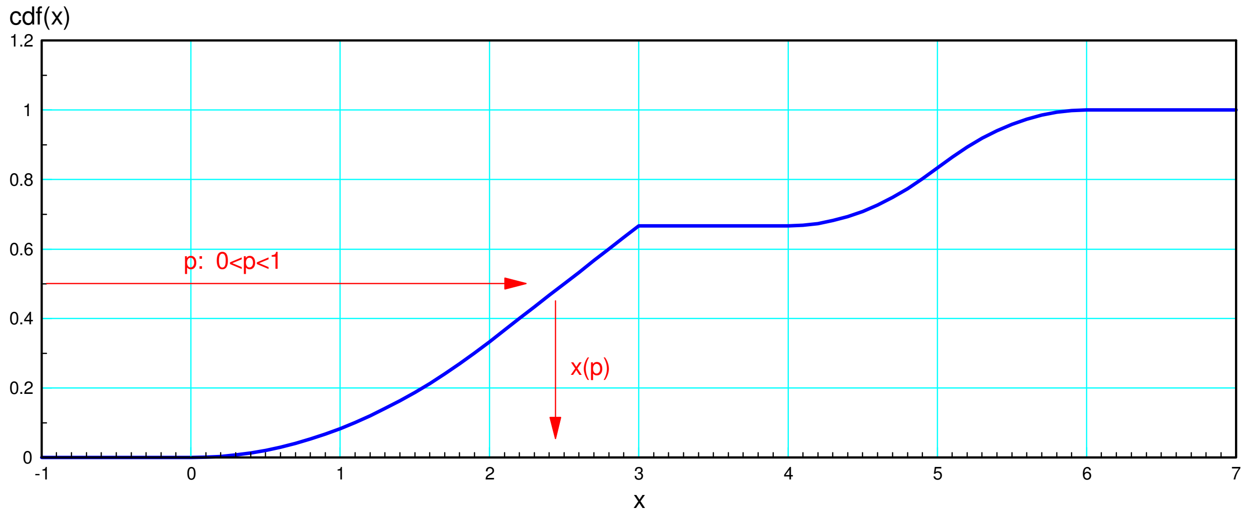
Note that the cdf must wind up at 1.000 (the probability of x being something is 100%)

In Matlab you can express this using if statements

```
function [ y ] = cdf( x )
if ( x < 0)
    y = 0;
elseif (x < 2)
    y = x*x/12;
elseif (x < 3)
    y = (x-2)/3 + 1/3;
elseif (x < 4)
     y = 2/3;
elseif (x < 5)
     y = (x-4)^2 /6 + 2/3;
elseif (x < 6)
    y = 1 - (6 - x)^2 / 6;
else y = 1;
end
```

which gives the cdf using the following code:

```
x = [-1:0.1:10]';
y = 0*x;

for i=1:length(x)
   y(i) = cdf(x(i));
   end

plot(x,y)
```



cdf: Note that the cdf is the integral of the pdf

To generate x,

- Determine the probability, p. Note that $0 < p < 1$
- The Y coordinate is p. Use the cdf to map p to x(p)

Solving backwards

$$p(x) = \begin{cases} 0 & x < 0 \\ \frac{x^2}{12} & 0 < x < 2 \\ \frac{x-2}{3} + \frac{1}{3} & 2 < x < 3 \\ \frac{2}{3} & 3 < x < 4 \\ \frac{(x-4)^2}{6} + \frac{2}{3} & 4 < x < 5 \\ 1 - \frac{(6-x)^2}{6} & 5 < x < 6 \\ 1 & 6 < x \end{cases}$$

becomes

$$x(p) = \begin{cases} \sqrt{12p} & 0 < p < 1/3 \\ 3p + 1 & 1/3 < p < 2/3 \\ 4 + \sqrt{6p - 4} & 2/3 < p < 5/6 \\ 6 - \sqrt{6(1-p)} & 5/6 < p < 1 \end{cases}$$

Example: in the range of 4 < x < 5

$$p = \frac{(x-4)^2}{6} + \frac{2}{3}$$

$$6p = (x-4)^2 + 4$$

$$(x-4)^2 = 6p - 4$$

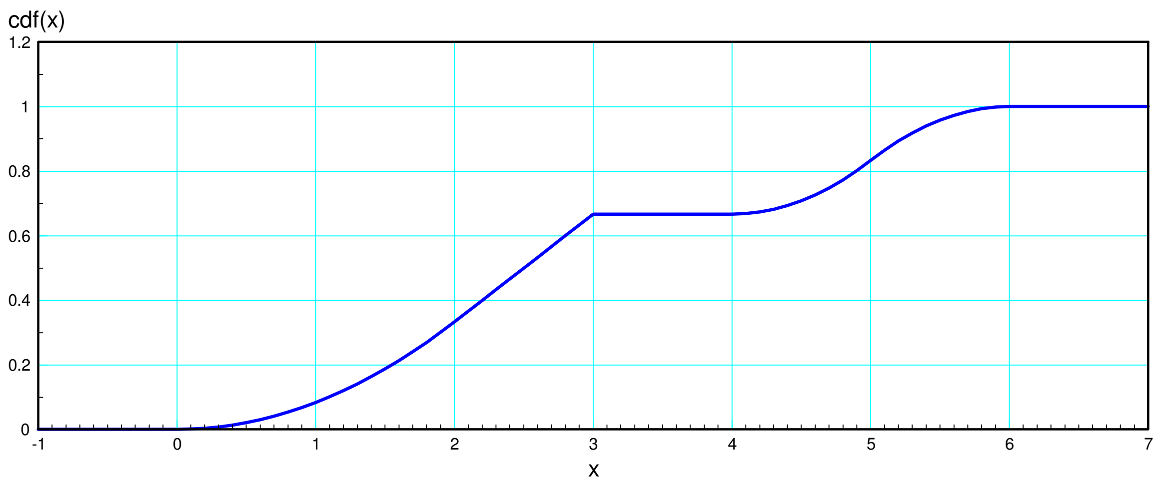
$$x = 4 + \sqrt{6p-4}$$

In matlab:

```
function [ x ] = pdf( p )
if ( p < 1/3)
    x = sqrt(12*p);
elseif (p < 2/3)
    x = 3*p+1;
elseif (p < 5/6)
    x = 4 + sqrt(6*p-4);
else
    x = 6 - sqrt(6*(1-p));
end
```

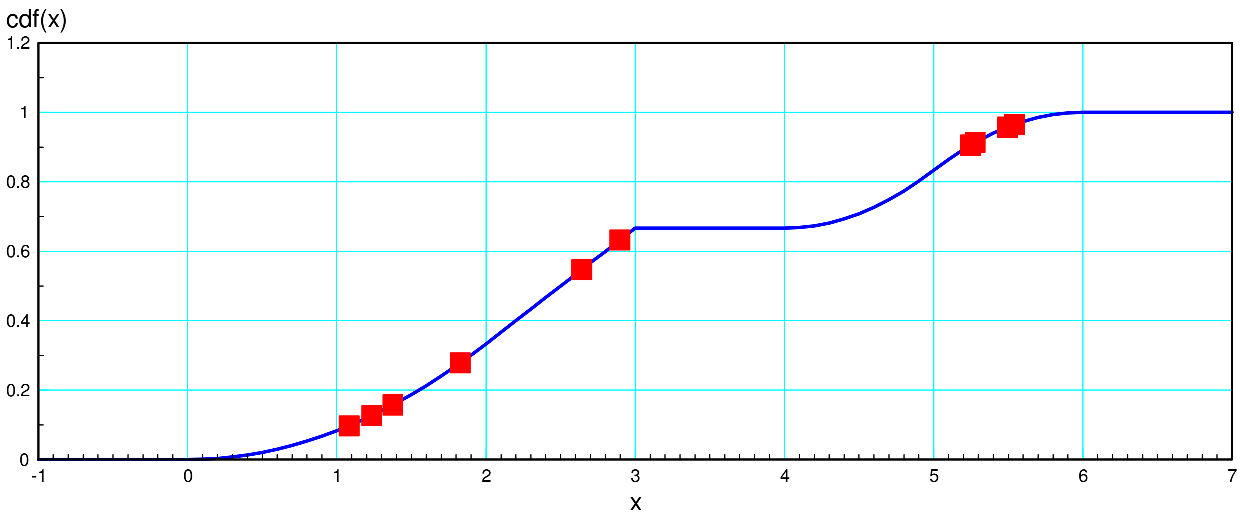
As a check, see if you get the graph as before:

```
p = [0:0.01:1]';
x = 0*p;
for i=1:length(p)
   x(i) = pdf(p(i));
   end
plot(x,p)
```



So now with this function, we can generate random values of x:

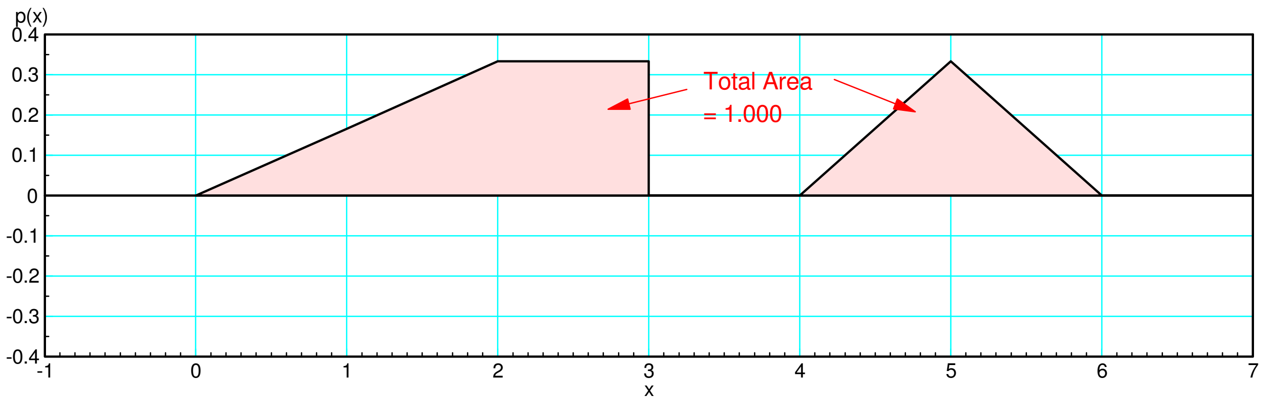
```
for i=1:10
   p = rand;
   x = pdf(p);
   disp([p,x])
   end

      p          x
   0.9058    5.2482
   0.1270    1.2344
   0.9134    5.2791
   0.6324    2.8971
   0.0975    1.0819
   0.2785    1.8281
   0.5469    2.6406
   0.9575    5.4951
   0.9649    5.5410
   0.1576    1.3753
```



Selecting random numbers for x (red dots) with a given cdf (blue line)

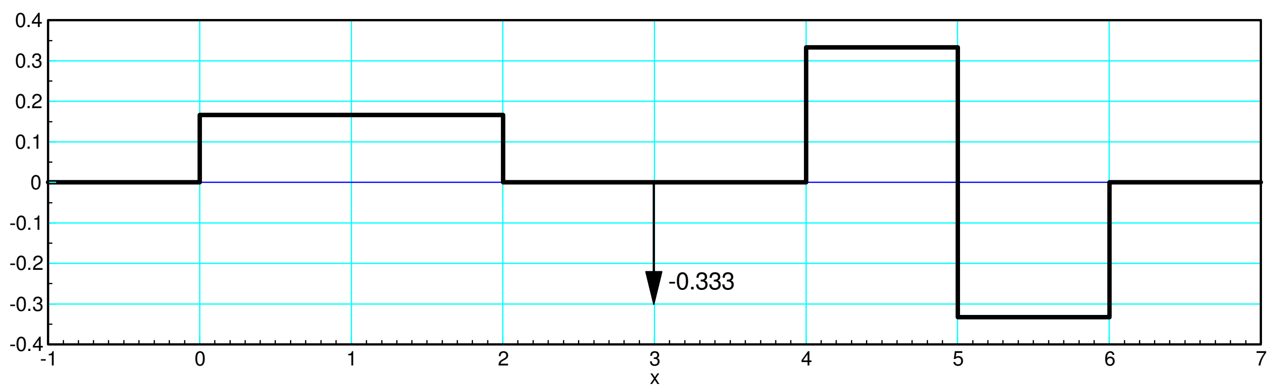
Moment Generating Function

Using the properties of LaPlace transforms, you can find the moment generating function (i.e. the LaPlace transform) for various functions. For example, consider the following pdf:



pdf: p(x)

These shapes don't have an obvious LaPlace transform, so take the derivative (the goal is to get to something we recognize, like a delta function):

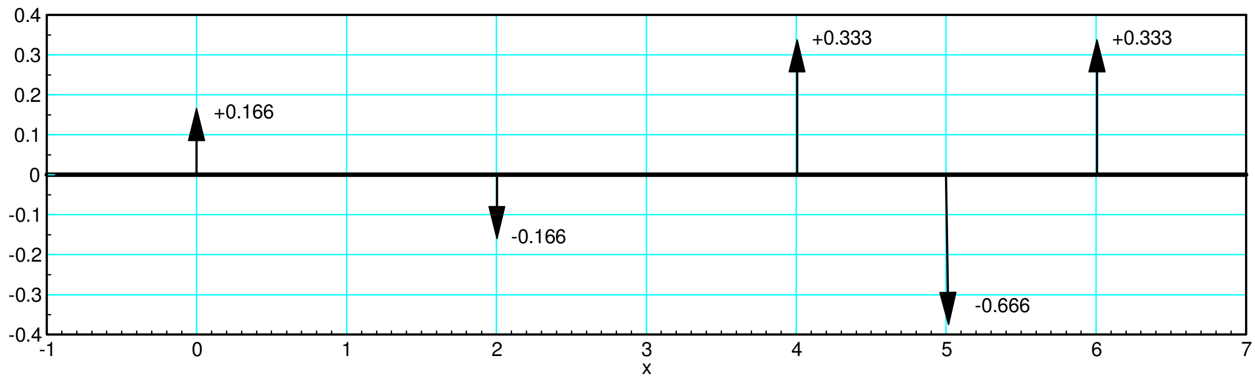


First derivative of pdf

The delta function has the LaPlace transform of an impulse (1), with a magnitude of -0.3333, delayed by 3 seconds

$$L(p') = -0.333e^{-3s}$$

Not take another derivative



Second derivative of pdf

The second derivative has the LaPlace transform of

$$L(p'') = 0.166 - 0.166e^{-2s} + 0.333e^{-4s} - 0.666e^{-5s} + 0.333e^{-6s}$$

To take the anti-derivative, you integrate (or in LaPlace, divide by s). This makes the moment generating function for p(x):

$$\psi(s) = L(p) + \frac{L(p')}{s} + \frac{L(p'')}{s^2}$$

$$\psi(s) = \left(\frac{-0.333e^{-3s}}{s}\right) + \left(\frac{0.166 - 0.166e^{-2s} + 0.333e^{-4s} - 0.666e^{-5s} + 0.333e^{-6s}}{s^2}\right)$$

Summary

Continuous probability functions can take on any value over a range. For such functions, LaPlace transforms are useful (aka *moment generating functions)*. When analyzing continuous distributions, the cdf is useful. With the cdf, you can generate random numbers with a given distrubution. LaPlace transforms are useful. With LaPlace tranforms, you can find the moments of a function as well as its mean and variance.