LaPlace Transforms

Table of LaPlace Transforms

Source: CRC Handbook of Mathematical Tables, CRC Press, 1964

Common LaPlace Transforms		
Name	Time: y(t) (t >0)	LaPlace: Y(s)
delta (impulse)	$\delta(t)$	1
unit step	1	$\left(\frac{1}{s}\right)$
unit ramp	t	$\left(\frac{1}{s^2}\right)$
unit parabola	t^2	$\left(\frac{2}{s^3}\right)$
	t^n	$\left(\frac{n!}{s^{n+1}}\right)$
decaying exponential	e^{-bt}	$\left(\frac{1}{s+b}\right)$
	$t\ e^{-bt}$	$\left(\frac{1}{(s+b)^2}\right)$
	$t^2\ e^{-bt}$	$\left(\frac{2}{(s+b)^3}\right)$
	$t^n\ e^{-bt}$	$\left(\frac{(n-1)!}{(s+b)^{n+1}}\right)$
	$2a \cdot e^{-bt}\cos(ct-\theta)u(t)$	$\left(\frac{a\angle\theta}{s+b+jc}\right) + \left(\frac{a\angle-\theta}{s+b-jc}\right)$

Transfer Functions and Differential Equations:

LaPlace transforms assume all functions are in the form of

$$y(t) = \begin{cases} a \cdot e^{st} & t > 0 \\ 0 & otherwise \end{cases}$$

This results in the derivative of y being:

$$\frac{dy}{dt} = s \cdot y(t)$$

This lets you convert differential equations into transfer funcitons and back.

Example 1: Find the transfer function that corresponds to the following differential equation:

$$\frac{d^3y}{dt^3} + 6\frac{d^2y}{dt^2} + 11\frac{dy}{dt} + 6y = 8\frac{dx}{dt} + 10x$$

Solution: Substitute 's' for $\frac{d}{dt}$

$$s^3Y + 6s^2Y + 11sY + 6Y = 8sX + 10X$$

Solve for Y

$$(s^3 + 6s^2 + 11s + 6)Y = (8s + 10)X$$

$$Y = \left(\frac{8s+10}{s^3+6s^2+11s+6}\right)X$$

The transfer function from X to Y is

$$\boxed{G(s) = \left(\frac{8s+10}{s^3+6s^2+11s+6}\right)}$$

Note: The transfer function is often called 'G(s)' since it is the gain from X to Y.

Example 2: Given the transfer function G(s),

$$G(s) = \left(\frac{8s+10}{s^3+6s^2+11s+6}\right)$$

determine the differential equation relating X and Y.

Solution: Go backwards. X and Y are related by:

$$Y = \left(\frac{8s+10}{s^3+6s^2+11s+6}\right)X$$

Cross multiply:

$$(s^3 + 6s^2 + 11s + 6)Y = (8s + 10)X$$

Note that 'sY' means 'the derivative of Y'

$$\boxed{\frac{d^3y}{dt^3} + 6\frac{d^2y}{dt^2} + 11\frac{dy}{dt} + 6y = 8\frac{dx}{dt} + 10x}$$

Sidelight: Fractional powers are not allowed in transfer functions.

- s^2Y means 'the second derivative of Y'.
- $s^{2.3}Y$ means 'the 2.3th derivative of Y'.

I have no idea what a 0.3 derivative is.

Solving Transfer Functions with Sinusoidal Inputs

Example 3: Find y(t) given

$$Y = \left(\frac{8s+10}{s^3+6s^2+11s+6}\right) X$$

and

$$x(t) = 3\cos(4t) + 5\sin(4t)$$

Solution: This is actually a phasor problem. From Euler's identity

$$e^{j4t} = \cos(4t) + j\sin(4t)$$

Take the real part and you get cosine. If you multiply by a complex number

$$(a+jb)\, e^{j4t} = a\cos(4t) - b\sin(4t) + j(stuff)$$

Take the real part and you get the phasor representation of a sine wave

$$a + jb \leftrightarrow a\cos(4t) - b\sin(4t)$$

Going back to the original problem

- G(s) is valid for all s
- X(s) only exists at s = j4, so

$$Y = \left(\frac{8s+10}{s^3+6s^2+11s+6}\right)_{s=j4} (3-j5)$$

$$Y = (-0.181 - j0.315)(3-j5)$$

$$Y = -2.120 - 0.040$$

This is the phasor representation for y(t)

$$y(t) = -2.120\cos(4t) + 0.040\sin(4t)$$

Example 4: Find y(t) if

$$x(t) = 6\cos(10t)$$

Same as before but s = j20 now:

$$Y = \left(\frac{8s+10}{s^3+6s^2+11s+6}\right)_{s=j10} (6+j0)$$

$$Y = (-0.067 - j0.034)(6+j0)$$

$$Y = -0.404 - j0.202$$

which is the phasor representation for

$$y(t) = -0.404\cos(10t) + 0.202\sin(10t)$$

Example 5: Find y(t) if

$$x(t) = 3\cos(4t) + 5\sin(4t) + 6\cos(10t)$$

Solution: Treat this as two separate problems

- x(t) = 3 cos(4t) + 5 sin(4t)
- x(t) = 6 cos(10t)

The total input is the sum of the two x(t)'s. The total output is the sum of the two y(t)'s

$$y(t) = -2.120\cos(4t) + 0.040\sin(4t) - 0.404\cos(10t) + 0.202\sin(10t)$$

Solving Transfer Functions with Step Inputs

There are several ways to do this. My preference is to use a table

Common LaPlace Transforms		
Name	Time: y(t)	LaPlace: Y(s)
delta (impulse)	$\delta(t)$	1
unit step	u(t)	$\frac{1}{s}$
	$a \cdot e^{-bt}u(t)$	$\frac{a}{s+b}$
	$2a \cdot e^{-bt}\cos(ct-\theta)u(t)$	$\left(\frac{a\angle\theta}{s+b+jc}\right)+\left(\frac{a\angle-\theta}{s+b-jc}\right)$

Example: Find the impulse response of

$$G(s) = \left(\frac{5}{s+3}\right)$$

Solution: Translating:

$$Y = \left(\frac{5}{s+3}\right)X$$

$$Y = \left(\frac{5}{s+3}\right)(1)$$

From the above table:

$$y(t) = 5e^{-3t}u(t)$$

Example: Find the step response of

$$G(s) = \left(\frac{5}{s+3}\right)$$

Solution: Translating:

$$Y = \left(\frac{5}{s+3}\right)\left(\frac{1}{s}\right) = \left(\frac{5}{s(s+3)}\right)$$

Here we have a problem: the above table doesn't include this type of function. Using partial fractions, however, you can turn it into something that is in the table

$$\left(\frac{5}{s(s+3)}\right) = \left(\frac{A}{s}\right) + \left(\frac{B}{s+3}\right)$$

Near s = 0, both sides go to infinity. The 'B' term is insignificant near s=0, so ignore it. Then

$$\left(\frac{5}{s(s+3)}\right)_{s\to 0} = \left(\frac{A}{s}\right)_{s\to 0}$$

Clearing out the 's' terms

$$\left(\frac{5}{s+3}\right)_{s \to 0} = A$$

$$A = 5/3$$

When s approaches -3, both sides again go to infinity. The 'A' term is finite and insignificant in this case, meaning

$$\left(\frac{5}{s(s+3)}\right)_{s \to -3} = \left(\frac{B}{s+3}\right)_{s \to -3}$$

Cancelling the (s+3) terms

$$\left(\frac{5}{s}\right)_{s \to -3} = B$$

$$B = -5/3$$

So,

$$Y = \left(\frac{5}{s(s+3)}\right) = \left(\frac{5/3}{s}\right) - \left(\frac{5/3}{s+3}\right)$$

Using the above table for each term:

$$\boxed{y(t) = \left(\frac{5}{3} - \frac{5}{3}e^{-3t}\right)u(t)}$$

Solving with Repeated Roots:

Option #1: Use partial fractions and an expanded table.

Exampe: Find the inverse LaPlace tranform for

$$Y(s) = \left(\frac{4}{(s+2)^2}\right)\left(\frac{1}{s}\right)$$

This expands as

$$Y(s) = \left(\frac{A}{s}\right) + \left(\frac{B}{(s+2)^2}\right) + \left(\frac{C}{s+2}\right)$$

Using the cover-up method

$$A = \left(\left(\frac{4}{(s+2)^2}\right)\left(\frac{1}{-}\right)\right)_{s=0} = 1$$

$$B = \left(\left(\frac{4}{-}\right)\left(\frac{1}{s}\right)\right)_{s=-2} = -2$$

$$C = \left(\frac{d}{ds}\left(\frac{4}{s}\right)\right)_{s=-2} = \left(\frac{-4}{s^2}\right)_{s=-2} = -1$$

giving

$$Y(s) = \left(\frac{1}{s}\right) + \left(\frac{-2}{(s+2)^2}\right) + \left(\frac{-1}{s+2}\right)$$

Now use a table to find the inverse transform of each term

$$y(t) = (1 - 2t\, e^{-2t} - e^{-2t})u(t)$$

Option #2: Change the problem so that the roots are no longer repeated. From a practical standpoint, there is almost no difference between

$$Y(s) = \left(\frac{4}{(s+2)(s+2)}\right)\left(\frac{1}{s}\right)$$

and

$$Y(s) = \left(\frac{4}{(s+2.01)(s+1.99)}\right)\left(\frac{1}{s}\right)$$

In the latter case, the roots are no longer repeated.

Solving with Complex Roots:

If you don't mind comlpex numbers, complex roots are no different than real roots: you just wind up with complex numbers in the partial fraction expansion.

The relevant entry in the LaPlace Transform table is

$$\left(\frac{a\angle\theta}{s+b+jc}\right) + \left(\frac{a\angle-\theta}{s+b-jc}\right) \Rightarrow 2a \cdot e^{-bt}\cos(ct-\theta)u(t)$$

Example: Find the y(t) given that

$$Y(s) = G \cdot U = \left(\frac{15}{s^2+2s+10}\right) \cdot \left(\frac{1}{s}\right)$$

Solution: Factoring Y(s)

$$Y(s) = \left(\frac{15}{(s)(s+1+j3)(s+1-j3)}\right)$$

Using partial fraction expansion:

$$Y(s) = \left(\frac{1.5}{s}\right) + \left(\frac{0.7906\angle-161.56^0}{s+1+j3}\right) + \left(\frac{0.7906\angle161.56^0}{s+1-j3}\right)$$

$$y(t) = 1.5 + 1.5812 \cdot e^{-t} \cdot \cos(3t + 161.56^0) \qquad \text{for } t>0$$

Properties of LaPlace Transforms

Properties of LaPlace Transforms

Let's start with some properties of LaPlace transforms:

Linearity: $af(t) + bg(t) \Leftrightarrow aF(s) + bG(s)$

Convolution: $f(t) * *g(t) \Leftrightarrow F(s) \cdot G(s)$

Differentiation: $\frac{dy}{dt} \Leftrightarrow sY - y(0)$

$$\frac{d^2y}{dt^2} \Leftrightarrow s^2Y - sy(0) - \frac{dy(0)}{dt}$$

Integration: $\int_0^t x(\tau)d\tau = \frac{1}{s}X(s)$

Delay $x(t-T) \Leftrightarrow e^{-sT}X(s)$

Proofs:

Linearity:

$$L(af(t) + bg(t)) = \int_{-\infty}^{\infty} (af(t) + bg(t)) \cdot e^{-st} \cdot dt$$

$$= \int_{-\infty}^{\infty} (af(t)) \cdot e^{-st} \cdot dt + \int_{-\infty}^{\infty} (bg(t)) \cdot e^{-st} \cdot dt$$

$$= a\int_{-\infty}^{\infty} f(t) \cdot e^{-st} \cdot dt + b\int_{-\infty}^{\infty} g(t) \cdot e^{-st} \cdot dt$$

$$= aF(s) + bG(s)$$

Convolution:

$$f(t) * *g(t) = \int_{-\infty}^{\infty} f(t-\tau) \cdot g(\tau) \cdot d\tau$$

$$L(f(t) * *g(t)) = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(t-\tau) \cdot g(\tau) \cdot d\tau \right) \cdot e^{-st} \cdot dt$$

$$= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(t-\tau) \cdot g(\tau) \cdot e^{-st} \cdot dt \right) \cdot d\tau$$

$$= \left(\int_{-\infty}^{\infty} f(t-\tau) \cdot e^{-st} \cdot dt \right) \cdot \left(\int_{-\infty}^{\infty} g(t) \cdot e^{-st} \cdot dt \right)$$

$$= F(s) \cdot G(s)$$

Differentiation:

$$L\left(\frac{dx}{dt}\right) = \int_{-\infty}^{\infty} \left(\frac{dx(t)}{dt}\right) \cdot e^{-st} \cdot dt$$

Assume causal (zero for t<0)

$$L\left(\frac{dx}{dt}\right) = \int_{0}^{\infty} \left(\frac{dx(t)}{dt}\right) \cdot e^{-st} \cdot dt$$

Integrate by parts.

$$(ab)' = a' \cdot b + a \cdot b'$$

$$\int a' \cdot b \cdot dt = ab - \int a \cdot b' \cdot dt$$

Let

$$a' = \frac{dx}{dt}$$

$$a = x$$

$$b = e^{-st}$$

then

$$L\left(\frac{dx}{dt}\right) = \int_{0}^{\infty} \left(\frac{dx(t)}{dt}\right) \cdot e^{-st} \cdot dt$$

$$= (x \cdot e^{-st})_0^{\infty} - \int_{-\infty}^{\infty} -s \cdot x(t) \cdot e^{-st} \cdot dt$$

$$= -x(0) + s \int_{-\infty}^{\infty} x(t) \cdot e^{-st} \cdot dt$$

$$= sX - x(0)$$

Integration:

$$L\left(\int_0^t x(\tau)\cdot d\tau\right) = \int_{-\infty}^{\infty}\left(\int_0^t x(\tau)\cdot d\tau\right)\cdot e^{-st}\cdot dt$$

Integrate by parts.

$$\int a\cdot b'\cdot dt = ab - \int a'\cdot b\cdot dt$$

Let

$$a = \int_0^t x(\tau)\cdot d\tau$$

$$b' = e^{-st}$$

then

$$a' = x$$

$$b = \tfrac{-1}{s}e^{-st}$$

$$L\left(\int_0^t x(\tau)\cdot d\tau\right) = \int_{-\infty}^{\infty}\left(\int_0^t x(\tau)\cdot d\tau\right)\cdot e^{-st}\cdot dt$$

$$= \left(\int_0^t x(\tau)\cdot d\tau\cdot \tfrac{-1}{s}e^{-st}\right)_{-\infty}^{\infty} - \int_{-\infty}^{\infty} x\cdot \tfrac{-1}{s}e^{-st}\cdot dt$$

Assuming the function vanishes at infinity

$$= \tfrac{1}{s}\int_{-\infty}^{\infty} x\cdot dt$$

$$= \left(\tfrac{1}{s}\right)X(s)$$

Time Delay

$$L(x(t-T)) = \int_{-\infty}^{\infty} x(t-T)\cdot e^{-st}\cdot dt$$

Do a change of variable

$$t - T = \tau$$

$$L(x(t-T)) = \int_{-\infty}^{\infty} x(\tau)\cdot e^{-s(\tau+T)}\cdot d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau)\cdot e^{-s\tau}\cdot e^{-sT}\cdot d\tau$$

$$= e^{-sT}\cdot\int_{-\infty}^{\infty} x(\tau)\cdot e^{-s\tau}\cdot d\tau$$

$$= e^{-sT}\cdot X(s)$$

Summary

When dealing with discrete probability funcitons or difference equations, z-transforms are useful. z-Transforms turn such equations into algebraic equations in z and turn convolution into multiplication.

When dealing with continuous probability functions of differential equations, LaPlace transforms are useful. LaPlace transforms turn such equations into algebraic equations in s and turn convolution into multiplication.

LaPlace transforms also allow you to go from the pdf to the cdf fairly easily:

$$cdf = \left(\frac{1}{s}\right) \cdot pdf$$