Geometric Distribution

Definitions:

- Uniform Distribution: The probability of each valid outcome is the same.
- Geometric Distribution: The number of Bernoulli trials until you get a success
- Pascal Distribution: The number of Bernoulli trials until you get r successes
- Geometric Distribution The number of times you roll a die until you get a one.
- The number of trips you make a trip with a car until something fails.
- The number of days until you an accident happens at work...

Distribution	description	pdf	mgf	mean	variance
Bernoulli trial	flip a coin obtain m heads	$p^m q^{1-m}$	$q + p/z$	p	p(1-p)
Binomial	flip n coins obtain m heads	$\binom{n}{m} p^m q^{n-m}$	$(q + p/z)^n$	np	np(1-p)
Hyper Geometric	Bernoulli trial without replacement	$\frac{\binom{A}{x}\binom{B}{n-x}}{\binom{A+B}{n}}$			
Uniform range = (a,b)	toss an n-sided die	$1/n \quad a \le m \le b$ $0 \quad otherwise$	$\left(\frac{1}{n}\right)\left(\frac{1+z+z^2+...+z^{n-1}}{z^b}\right)$	$\left(\frac{a+b}{2}\right)$	$\left(\frac{(b-a+1)^2-1}{12}\right)$
Geometric	Bernoulli until 1st success	$p\ q^{k-1}$	$\left(\frac{p}{z-q}\right)$	$\left(\frac{1}{p}\right)$	$\left(\frac{q}{p^2}\right)$
Pascal	Bernoulli until rth success	$\binom{k-1}{r-1} p^r q^{k-r}$	$\left(\frac{p}{z-q}\right)^r$	$\left(\frac{r}{p}\right)$	$\left(\frac{rq}{p^2}\right)$

Geometric Distribution:

A geometric distribution is one where you conduct a Bernoulli trial (think: flip a coin) until you get a success. The pdf for a geometric distribution is:

$$f(k) = \begin{cases} p\ q^{k-1} & k = 1, 2, 3 \\ 0 & otherwise \end{cases}$$

where 'p' is the probability of a success and x is the number of flips it takes before you get a success.

To see this, consider the following with tossing a coin. Assume the probability of a heads is 'p'. The probability of getting a heads on nth flip is the probability of getting n-1 tails followed by a heads:

- x=1: f() = p
- x = 2: f() = p q
- x = 3: f() = p q^2
- etc.

Note that the '1' in the notation means the game is over after the first success. You might guess that there will be more general distribution where you look for m successes. You'd be right...

The pdf for a geometric distribution looks like the following:

p = 0.9

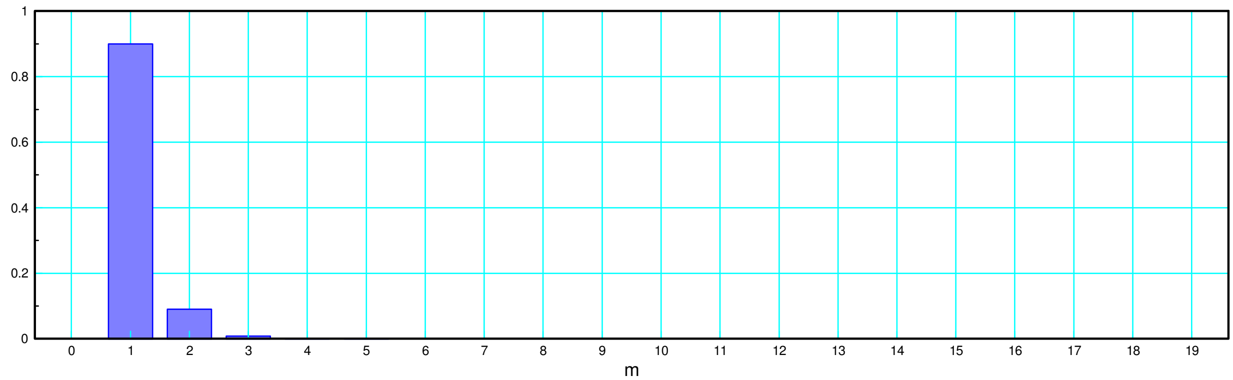
```
f = zeros(20,1);

p = 0.9;

for i=1:20
   f(i) = p * (1-p)^(i-1);
   end
```

Note that the probability that something happens is one:

```
-->sum(f)
    1.
```

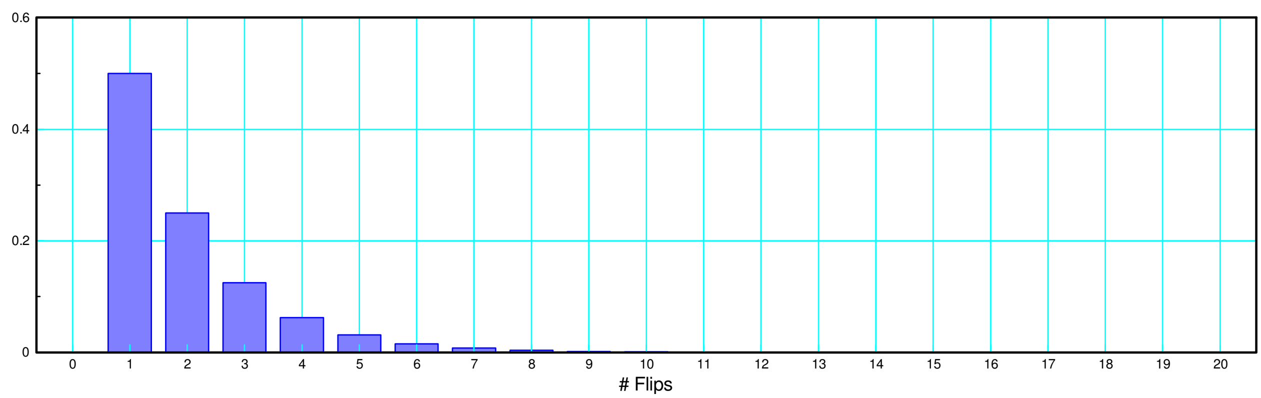
Plotting the pdf shows what the distribution looks like:

```
-->bar(f)
-->xlabel('Number of Flips')
```

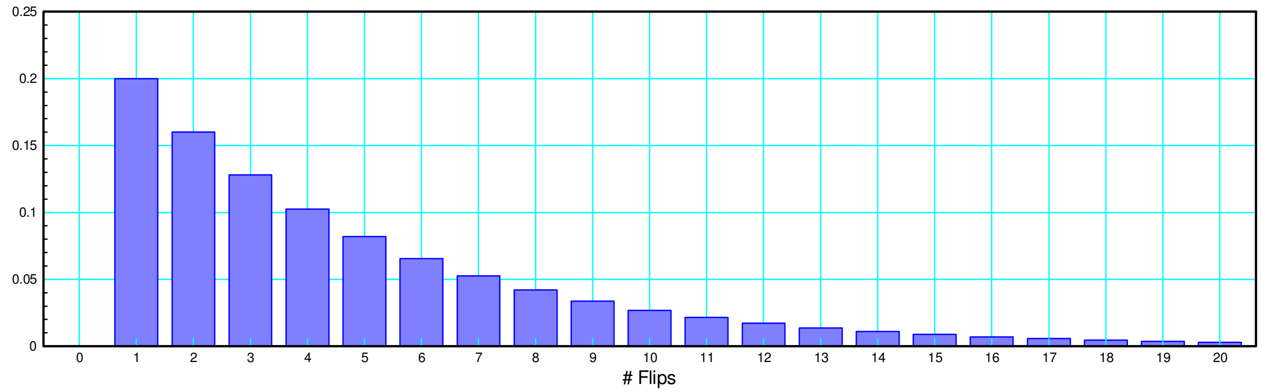


pdf for a geometric distribution with p = 0.9

Repeating for p = 0.5 and 0,2:



pdf for a geometric distribution with p = 0.5



pdf for a geometric distribution with p = 0.2

Note that for a geometric distribution, the probability of a success for each toss is the same. Examples of this would be:

- Tossing a coin until you get a heads
- Betting on 10-black in Roulette until you finally win
- Buying a lottery ticket each week until you finally win
- Trying to open a door with n keys where you replace the key after each trial and try again (and again and again..) This is called sampling with replacement.

Mean for a Geometric Distribution:

$$\mu = \sum_{k=1}^{\infty} k \cdot p \cdot q^{x-1}$$

$$\mu = p(1 + q + 2q^2 + 3q^3 + 4q^4 + ...)$$

Variance for a Geometric Distribution:

$$\sigma^2 = \sum_{k=1}^{\infty} (k-\mu)^2 \cdot p \cdot q^{k-1}$$

You can kind of see that we need a better tool. That will be moment generating functions (coming in a lecture shortly.). The net result is going to be....

$$\mu = \frac{1}{p}$$

$$\sigma^2 = \frac{q}{p^2}$$

Moment Generating Function for an Exponential Distribution:

The time-series (where m means time) is

$$x(k) = q \cdot x(k-1)$$

$$x(1) = p$$

Taking the z-transform

$$x(k) = q \cdot x(k-1) + p\ \delta(k-1)$$

$$X = q\ z^{-1} X + p\ z^{-1}$$

Solve for X

$$(z-q)X = p$$

$$\psi = \left(\frac{p}{z-q}\right)$$

Using this, you can find moments as well as the mean and variance[1]

Zeroth Moment:

- m_o must be 1.000 to be a valid pdf (all probabilities add to 1)

$$m_0 = \psi(z=1) = \left(\frac{p}{z-q}\right)_{z=1} = \left(\frac{p}{1-q}\right) = \left(\frac{p}{p}\right) = 1$$

1st-Moment (mean)

- The first moment is the mean

$$m_1 = -\frac{d}{dz}(\psi(z))_{t=1} = -\psi'(z=1)$$

$$m_1 = -\frac{d}{dz}\left(\frac{p}{z-q}\right)_{z=1} = -\left(\frac{-p}{(z-q)^2}\right)_{t=1} = \left(\frac{p}{(1-q)^2}\right) = \left(\frac{p}{p^2}\right) = \left(\frac{1}{p}\right)$$

$$\mu = m_1 = \left(\frac{1}{p}\right)$$

Second Moment:

- $m_2 = \frac{d^2}{dz^2}(\psi(z))_{z=1} = \psi''(z=1)$
- $m_2 = \frac{d}{dz}\left(\frac{p}{(z-q)^2}\right) = \left(\frac{-2p(z-q)}{(z-q)^4}\right)_{z=1} = \left(\frac{2p(1-q)}{(1-q)^4}\right) = \left(\frac{2p^2}{p^4}\right) = \left(\frac{2}{p^2}\right)$

[1] Probability and Statistics, Morris DeGroot

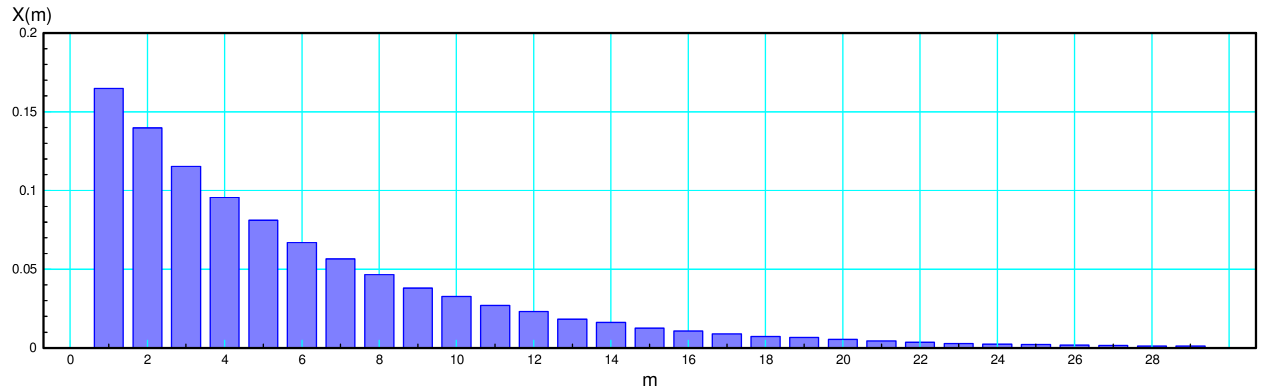
Variance:

- $\sigma^2 = m_2 - m_1^2$
- $\sigma^2 = \left(\frac{2}{p^2}\right) - \left(\frac{1}{p}\right)^2 = \left(\frac{1}{p^2}\right)$
- $\sigma^2 = \left(\frac{q}{p^1}\right)$ *actual variance: not sure why I'm off by q*

That was a *lot* easier than applying the definition. z-transforms are really useful

Matlab Example: Toss a die until you roll a 6 (p = 1/6). Determine the mean and standard deviation after 10,000 games

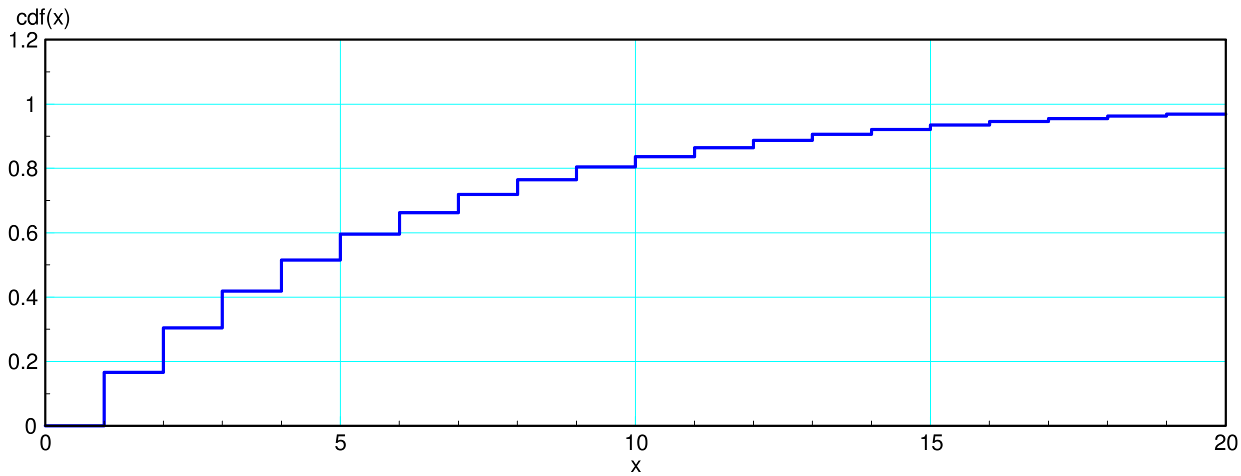
```
N = 1e5;
X = zeros(100,1);
p = 1/6;
q = 1-p;

for i=1:N

   n = 1;

   while(rand > p)
       n = n + 1;
   end

   X(n) = X(n) + 1;
end

X = X / N;

M = [1:100]';
x = sum(M .* X);
s2 = sum(X .* (M-x).*(M-x));

disp([x,1/p])
disp([s2,q/(p*p)])


          Sim        Calc
x        6.0179     6.0000
var     30.0712    30.0000
```



Experimental pdf for tossing a die until you roll a 6

The cdf is the integral (sum) of the pdf from 0 to x:

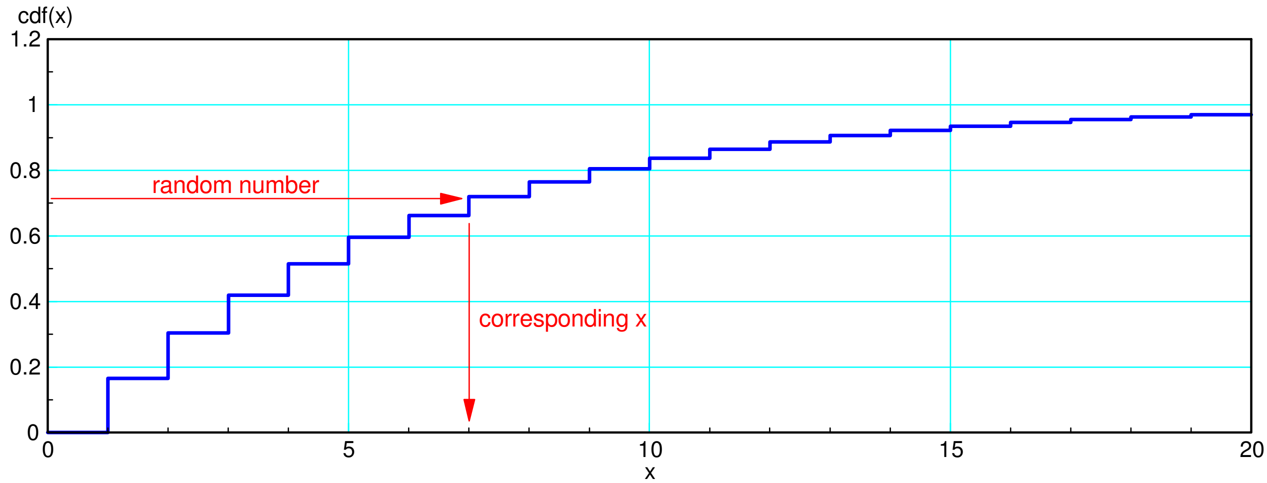
```
cdf = 0*X;
for i=1:length(cdf)
   cdf(i) = sum(pdf(1:i));
   end
```



Experimental cdf for a geometric distribution

The cdf is a more useful way of generating x

- Pick a random number in the interval of (0, 1)
  - This is the y-coordinate
- Find the corresponding x



You can also compute the cdf using z-transforms (with a lot less work). The cdf is the integral of the pdf:

$$cdf = pdf \cdot \left(\frac{z}{z-1}\right)$$

or

$$cdf = \left(\frac{p}{z-q}\right)\left(\frac{z}{z-1}\right)$$

$$= \left(\frac{p}{(z-q)(z-1)}\right)z = \left(\frac{1}{z-1} + \frac{-1}{z-q}\right)z$$

$$cdf = 1 - q^x$$

Solving backwards

$$x = ceil\left(\frac{\ln(1-cdf)}{\ln(q)}\right)$$

To find x:

- Pick a random number in the range of (0, 1)
- Convert to x using the above formula

Expected Return

The mean of a distribution is important. For example, suppose you're playing a game:

- It costs $N to play the game.
- To play the game, you roll a 6-sided die over and over again until you roll a 1.
- The payout is the number of times you roll the dice.

In this game

$$p = \frac{1}{6}$$

and the mean is

$$\mu = \frac{1}{p} = 6$$

This means you expect to get paid $6 on average every time you play the game.

The expected return is

- The expected return ($6),
- Minus the cost to play the game ($N).

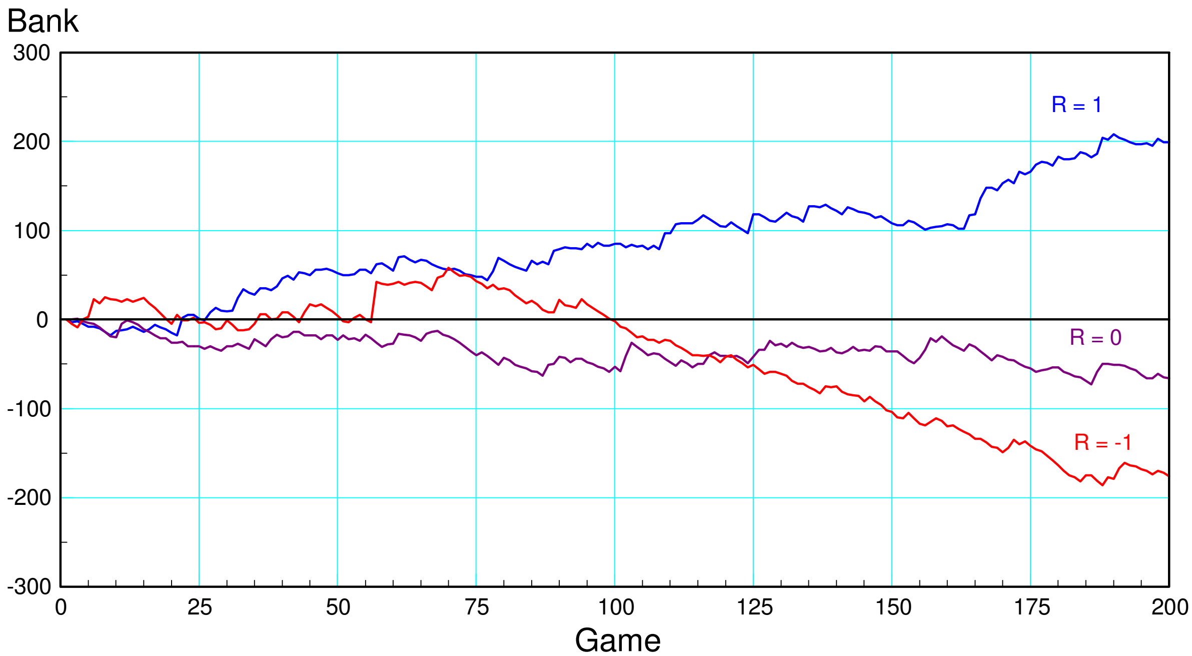
If the expected return is positive, you should keep playing this game (you expect to make money). If the expected return is negative, you should avoid this game or stop playing it: you expect to lose money.

For example, the money you make when playing this game when

- N = 5 (you expect to make money)
- N = 6 (you expect to break even), and
- N = 7 (you expect to lost money)

is simulated in Matlab for 200 games.

```
Bank = zeros(200,1);
Cost = 5;
Bank(1) = 100;
for k=2:200
    n = 1;
    while(rand > 1/6)
        n = n + 1;
    end
    Bank(k) = Bank(k-1) + n - Cost;
end
B5 = Bank;
t = [1:200]';
plot(t,Bank);
```



Simulated earnings when the expected return is +1 (blue), 0 (purple), and -1 (red)

Note

- When the expeceted return is positive, over the long haul, you should be up
- When the expeceted return is zero, you should remain close to zero
- When the expected return is negative, over the long haul, you should be down

Likewise, the mean of a pdf is important:

- It tells you your expected return if playing this game,
- It tells you when too expensive to play (negative expected return) or when it's a bargain (positive expected return).

State Lotteries

Along these lines, let's look at state lotteries. In North Dakota, you can buy lottery tickets online. This means that, according to federal law and federal trade agreements, online gambling is legal in North Dakota. North Dakota also has laws prohibiting online gambling. That means that online gambling is *not* legal. You figure it out.

Regardless, one of the lottery games North Dakota offers is 2-by-2. In this game

- You pay $1 for a ticket.
- Each ticket has two red numbers (1..26) and two white numbers (1..26)

The payout is

Match	Prize (MWRF) x	Combinations M	Return x * M/N
4 Numbers	22,000	1	0.2083
3 Numbers	100	96	0.0909
2 Numbers	3	2,856	0.0811
1 Number	1	13,248	0.1254
		Total	0.5057

The total number winning numbers

$$N = \binom{26}{2}\binom{26}{2} = 105625$$

Number of combinations with 4 winning numbers

$$M = \binom{2}{2}\binom{24}{0} \cdot \binom{2}{2}\binom{24}{0} = 1$$

Number of combinations with 3 winning numbers

$$M = 2 \cdot \binom{2}{2}\binom{24}{0} \cdot \binom{2}{1}\binom{24}{1} = 96$$

Number of combinations with 2 winning numbers

$$M = 2 \cdot \binom{2}{2}\binom{24}{0} \cdot \binom{2}{0}\binom{24}{2} + 1 \cdot \binom{2}{1}\binom{24}{1} \cdot \binom{2}{1}\binom{24}{1} = 2856$$

Number of combinations with 1 winning number

$$M = 2 \cdot \binom{2}{1}\binom{24}{1} \cdot \binom{2}{0}\binom{24}{2} = 13,248$$

The expected return for a $1 bet is $0.5057[2]. This means you lose on average 49 cents every time you bet a dollar.

A return of 50 cents for every dollar bet is actually a very typical return for state lotteries. On the principle that

- You should consider playing games with a positive expected return, and
- You should avoid games with a negative expected return (like state lotteries)

you would expect that no-one would ever play a state lottery. Yet, people do. Why?

One thought is that money isn't linear:

- Losing $1 on a lottery ticket won't change your life, but
- Winning the lottery can change your life.

With $22,000, you can pay off your credit cards, pay off your car, etc. With $22,000, you can make a new start.

[2] Except on Tuesdays where the payout doubles. After taxes, it's still a losing bet, however.

This line of though would suggest that poor people (where $22,000 is a life-changing sum) would be more inclined to play the state lottery than rich people (where $22,000 isn't that big of deal). That's exactly what you see - so maybe it *is* the case that money isn't linear. At least if you're poor.

Gauss' Dilemma:

This leads to a fairly famous casino game: *Gauss' Dilemma*. This is a casino game which

- No-one will play because you (almost) always lose, and
- No-one will offer because the expected winnings are infinite.

The game goes as follows:

- Pay some amount to play, such as $100, and start with $1 in the pot.
- Toss a coin. If it comes up tails, double the pot.
- Keep playing until the coin comes up heads. Once that happens, the game ends and you collect your winnings.

Should you play this game?

To determine if this is a fair game, let's compute the expected return. This is a geometric distribution with the probability density function being

# Tosses (m)	1	2	3	4	5	6
Probability (p)	1/2	1/4	1/8	1/16	1/32	1/64
Pot (x)	1	2	4	8	16	32
Winnings (p*x)	1/2	1/2	1/2	1/2	1/2	1/2

The expected winnings are the cost to play (-$100) plus the sum of the pots times their probabilities:

$$E = \sum p(m) \cdot x(m) - 100$$

$$E = \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} + \ldots - 100$$

$$E = \infty$$

With infinite expected winnings, this sounds like a good game to play.

To check this out, let's run a Monte-Carlo simulation with ten games:

```
% Gauss' Dilemma

N = 10;
Winnings = 0;
p = 0.5;

for i=1:N

   Pot = 1;
```

```
    while(rand > p)
        Pot = Pot * 2;
    end

    Winnings = Winnings + Pot - 100;
end

Winnings / N

-98.2
```

Each time you play, you lose on average $98.2

The expected winnings are infinite - so maybe we didn't play long enough. If you increase the number of games to 1000, the average winnings are still negative:

```
Winnings / N  =  -95.0180
```

With 1 million games, the average winnings remain negative:

```
Winnings / N  =  -89.7185
```

Likewise, it's a really bad game to play: Monte-Carlo simulations show that every player loses money. However, with expected winnings of infinity, it's also a really bad game to offer. This makes this a game where, essentially,

- Every player loses money and
- Every casino loses money

Hence the name Gauss' Dilemma

Summary

Geometric distributions are when you keep playing a game until an even happens, such as rolling a 1 on a 6-sided die. These distributions are infinite in length - meaning that Monte-Carlo simulations can take a long time to execute as you keep waiting for an event to happen. This is where the cdf is really useful: the cdf provides a closed-for solution for a variable with a geometric distribution that requires no iteration.