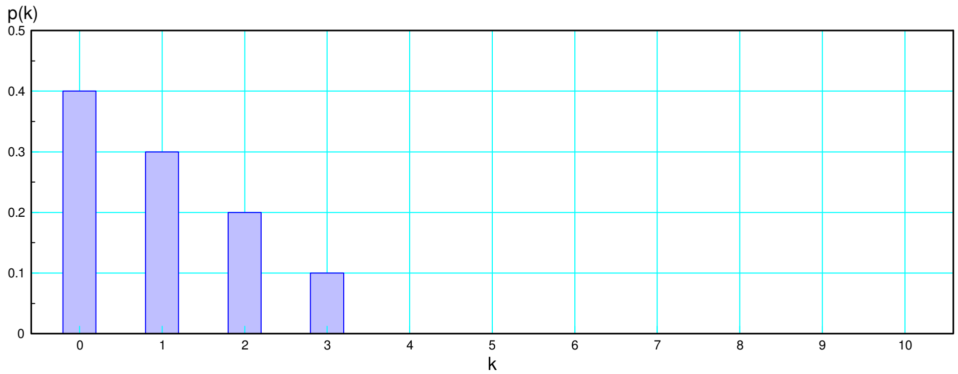
z-Transforms



Discrete pdf with a z-transform of $\psi(z) = 0.4 + 0.3\left(\frac{1}{z}\right) + 0.2\left(\frac{1}{z^2}\right) + 0.1\left(\frac{1}{z^3}\right)$

z-Transform:

z-transforms are similar to LaPlace transforms - only they apply to discrete-time systems (or discrete probability density functions). The z-transform is defined as

$$X(z) = \sum_{n=-\infty}^{\infty} x_n \cdot z^{-n}$$

Hence, the above figure which has the pdf of

$$pdf = 0.4\ \delta(k) + 0.3\ \delta(k-1) + 0.2\ \delta(k-2) + 0.1\ \delta(k-3)$$

has a z-transform (also known as the moment generating function) of

$$\psi(z) = 0.4 + 0.3\left(\frac{1}{z}\right) + 0.2\left(\frac{1}{z^2}\right) + 0.1\left(\frac{1}{z^3}\right)$$

The assumption behind the z-transform is that all functions are in the form of

$$y(k) = z^k$$

When you move forward in time by one sample, you multiply by 'z'.

$$y(k+1) = z^{k+1} = z \cdot z^k = z \cdot y(k)$$

'zy' is read as 'the next value of y'. Likewise, the z-transform converts difference equations into algebraic equations in 'z'.

z-Transform Properties: (www.wikipedia.com)

The z-transform is defined as

$$X(z) = \sum_{n=-\infty}^{\infty} x_n \cdot z^{-n}$$

Properties:

Linearity:

$$Z(ax_n + by_n) = aX(z) + bY(z)$$

Proof: The z-transform is

$$Z(ax_n + by_n) = \sum_{n=-\infty}^{\infty} (ax_n + by_n) \cdot z^{-n}$$

$$Z(ax_n + by_n) = \left(a \sum_{n=-\infty}^{\infty} x_n \cdot z^{-n} \right) + \left(b \sum_{n=-\infty}^{\infty} y_n \cdot z^{-n} \right)$$

$$Z(ax_n + by_n) = aX(z) + bY(z)$$

Time Shifting:

$$Z(x_{n-k}) = z^{-k} \cdot X(z)$$

Proof:

$$Z(x_{n-k}) = \sum_{n=-\infty}^{\infty} x_{n-k} \cdot z^{-n}$$

Let m = n-k

$$Z(x_{n-k}) = \sum_{m=-\infty}^{\infty} x_m \cdot z^{-(m+k)}$$

$$Z(x_{n-k}) = \sum_{m=-\infty}^{\infty} x_m \cdot z^{-m} \cdot z^{-k}$$

$$Z(x_{n-k}) = z^{-k} \cdot \left(\sum_{m=-\infty}^{\infty} x_m \cdot z^{-m} \right)$$

$$Z(x_{n-k}) = z^{-k} \cdot X(z)$$

Multiplying by 1/z means delay the signal by one.

Convolution:

$$Z(x_n ** y_n) = X(z) \cdot Y(z)$$

Proof:

$$Z\left(\sum_{k=-\infty}^{\infty} x_k \cdot y_{n-k}\right) = \sum_{n=-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} x_k \cdot y_{n-k}\right) \cdot z^{-n}$$

Change the order of summation:

$$= \sum_{k=-\infty}^{\infty} \left(\sum_{n=-\infty}^{\infty} x_k \cdot y_{n-k}\right) \cdot z^{-n}$$

$$= \left(\sum_{k=-\infty}^{\infty} x_k \left(\sum_{n=-\infty}^{\infty} y_{n-k}\right)\right) \cdot z^{-n}$$

Let m = n-k

$$= \left(\sum_{k=-\infty}^{\infty} x_k \left(\sum_{n=-\infty}^{\infty} y_m\right)\right) \cdot z^{-(m+k)}$$

$$= \left(\left(\sum_{k=-\infty}^{\infty} x_k \cdot z^{-k}\right)\left(\sum_{n=-\infty}^{\infty} y_m \cdot z^{-m}\right)\right)$$

$$= X(z) \cdot Y(z)$$

Convolution is the biggie - it turns convolution into multiplication.

Table of z-Transforms:

function	y(k) (k > 0)	Y(z)
delta	$\delta(k) = \begin{cases} 1 & k=0 \\ 0 & otherwise \end{cases}$	1
unit step	$u(k) = 1$	$\left(\frac{z}{z-1}\right)$
ramp	k	$\left(\frac{z}{(z-1)^2}\right)$
parabola	k^2	$\left(\frac{z(z+1)}{(z-1)^3}\right)$
cubic	k^3	$\left(\frac{z(z^2+4z+1)}{(z-1)^4}\right)$
decaying exponential	a^k	$\left(\frac{z}{z-a}\right)$
	$\left(\frac{k}{1!}\right)a^{k-1}$	$\left(\frac{z}{(z-a)^2}\right)$
	$\left(\frac{k(k-1)}{2!}\right)a^{k-2}$	$\left(\frac{z}{(z-a)^3}\right)$
	$\left(\frac{k(k-1)(k-2)}{3!}\right)a^{k-3}$	$\left(\frac{z}{(z-a)^4}\right)$
damped sinewave	$2b \cdot a^k \cdot \cos(k\theta + \phi) \cdot u(k)$	$\left(\frac{(b\angle\phi)z}{z-(a\angle\theta)}\right) + \left(\frac{(b\angle-\phi)z}{z-(a\angle-\theta)}\right)$

Proof: Delta Function. This is sort-of the definition of z-transform:

$$X(z) = \sum_{n=-\infty}^{\infty} x_n \cdot z^{-n} = \ldots + x_0 \cdot z^0 + x_1 \cdot z^1 + x_2 \cdot z^2 + \ldots$$

If x(k) is a delta function:

$$X(z) = 1$$

Proof: Unit Step. Using a table:

	z^2	z^1	z^0	z^{-1}	z^{-2}	z^{-3}	z^{-4}
X(z)	0	0	1	1	1	1	1
z^{-1} X(z)	0	0	0	1	1	1	1
subtract							
$\left(1 - \frac{1}{z}\right)X(z)$	0	0	1	0	0	0	0

so

$$X(z) = \frac{1}{\left(1-\frac{1}{z}\right)} = \left(\frac{z}{z-1}\right)$$

Proof: Decaying Exponential. Using a table:

	z^2	z^1	z^0	z^{-1}	z^{-2}	z^{-3}	z^{-4}
X(z)	0	0	1	a	a^2	a^3	a^4
a * z^{-1} X(z)	0	0	0	a	a^2	a^3	a^4
subtract							
$(1-\frac{a}{z})X(z)$	0	0	1	0	0	0	0

so

$$X(z) = \left(\frac{1}{1-\frac{a}{z}}\right) = \left(\frac{z}{z-a}\right)$$

Solving Functions in the z-Domain

Problem 1: Find the step response of

$$Y = \left(\frac{0.2z}{(z-0.9)(z-0.5)}\right)X$$

i) Replace X(z) with the z-transform of a step

$$Y = \left(\frac{0.2z}{(z-0.9)(z-0.5)}\right)\left(\frac{z}{z-1}\right)$$

ii) Use partial fractions to expand this. Note, however, that the table entries have a 'z' in the numerator. So, factor this out first then take the partial fraction expansion

$$Y = \left(\frac{0.2z}{(z-1)(z-0.9)(z-0.5)}\right)z$$

$$Y = \left(\left(\frac{4}{z-1}\right) + \left(\frac{-4.5}{z-0.9}\right) + \left(\frac{0.5}{z-0.5}\right)\right)z$$

Multiply through by z

$$Y = \left(\left(\frac{4z}{z-1}\right) + \left(\frac{-4.5z}{z-0.9}\right) + \left(\frac{0.5z}{z-0.5}\right)\right)$$

iii) Now apply the table entries

$$y(k) = 4 - 4.5 \cdot (0.9)^k + 0.5 \cdot (0.5)^k \qquad k >= 0$$

Problem 2: Repeated Poles: Assume the pdf for a system is

$$PDF = \left(\frac{0.004}{(z-0.9)(z-0.8)^2}\right)$$

Find the CDF.

Solution: The CDF is the integral of the PDF. In the z-domain, integration is equivalent to multiplying by $\left(\frac{z}{z-1}\right)$

$$CDF = \left(\frac{0.004}{(z-0.9)(z-0.8)^2}\right)\left(\frac{z}{z-1}\right)$$

This isn't in the table of z-transforms, so do a partial fraction expansion. When you have repeated poles, there will also be terms for each lower power of the repeated pole. This means the solution will look like (after pulling out a z term)

$$\left(\frac{0.004}{(z-0.9)(z-0.8)^2}\right)\left(\frac{1}{z-1}\right)z = \left(\left(\frac{a}{z-0.9}\right) + \left(\frac{b}{(z-0.8)^2}\right) + \left(\frac{c}{z-0.8}\right) + \left(\frac{d}{z-1}\right)\right)z$$

The terms {a, b, d} can be found using the cover-up method

$$a = \left(\frac{0.004}{(z-0.8)^2(z-1)}\right)_{z=0.9} = -4$$

$$b = \left(\frac{0.004}{(z-0.9)(z-1)}\right)_{z=0.8} = 0.2$$

$$d = \left(\frac{0.004}{(z-0.9)(z-0.8)^2}\right)_{z=1} = 1$$

There are several ways to find c. My preference is placing over a common denominator and matching coefficients

$$\left(\frac{0.004}{(z-0.9)(z-0.8)^2(z-1)}\right) = \left(\frac{a}{z-0.9}\right)\left(\frac{(z-0.8)^2(z-1)}{(z-0.8)^2(z-1)}\right) + \left(\frac{b}{(z-0.8)^2}\right)\left(\frac{(z-0.9)(z-1)}{(z-0.9)(z-1)}\right) +$$
$$\left(\frac{c}{z-0.8}\right)\left(\frac{(z-0.9)(z-0.8)(z-1)}{(z-0.9)(z-0.8)(z-1)}\right) + \left(\frac{d}{z-1}\right)\left(\frac{(z-0.9)(z-0.8)^2}{(z-0.9)(z-0.8)^2}\right)$$

The numberator is

$$0.004 = a\cdot(z-0.8)^2(z-1) + b\cdot(z-0.9)(z-1) + c\cdot(z-0.9)(z-0.8)(z-0.7)$$
$$+d(z-0.9)(z-0.8)^2$$

All terms have to balance. Picking the z^3 term (other terms give the same result):

$$0 = a + c + d$$

$$c = -5$$

meaning

$$CDF = \left(\left(\frac{4}{z-0.9} \right) + \left(\frac{0.2}{(z-0.8)^2} \right) + \left(\frac{-5}{z-0.8} \right) + \left(\frac{1}{z-1} \right) \right) z$$

Multuply through by z and take the inverse z-transform.

$$cdf(k) = \left(4(0.9)^k + 0.2k(0.8)^k - 5(0.8)^k + 1 \right) u(k)$$

Problem 3: Complex Poles: Find the step response of a system with complex poles:

$$Y = \left(\frac{0.2z}{\left(z-0.9\angle 10^0\right)\left(z-0.9\angle -10^0\right)} \right) X$$

i) Replace X with its z-transforrm (a unit step)

$$Y = \left(\frac{0.2z}{\left(z-0.9\angle 10^0\right)\left(z-0.9\angle -10^0\right)} \right) \left(\frac{z}{z-1} \right)$$

ii) Factor our a z and use partial fractions

$$Y = \left(\left(\frac{5.355}{z-1} \right) + \left(\frac{2.98\angle 153.97^0}{z-0.9\angle 10^0} \right) + \left(\frac{2.98\angle -153.97^0}{z-0.9\angle -10^0} \right) \right) z$$

iii) Convert back to time using the table of z-transforms

$$(k) = 5.355 + 4.859 \cdot (0.9)^k \cdot \cos\left(10^0 \cdot k - 153.97^0\right) \qquad k >= 0$$

Note that in the s-domain, rectangular coordinates are more convenient

- The real part of s tells you the rate at which the exponential decays
- The complex part of s tells you the frequency of oscillations.

In the z-domain, polar coordinates are more convenient

- The amplitude of z tells you the rate at which the signal decays
- The angle of z tells you the frequency of oscillation.

In this case,

- The signal decays by 10% each sample (0.9)k
- The phase changes by 10 degrees each sample, or requires 36 samples per cycle

Time Value of Money

As a sidelight, you can also solve time-value of money problems using z-transforms.

Assume you borrow \$100,000 for a house. How much do you have to pay each month to pay off the loan in 10 years? Assume 6% interest per year (0.5% per month).

Solution: Let x(k) be how much money you owe today. The amount you owe next month, x(k+1), is

$$x(k+1) = 1.005x(k) - p + X(0) \cdot \delta(k)$$

where 'p' is your monthly payment. Converting to the z-domain

$$zX - X(0) = 1.005X - p\left(\frac{z}{z-1}\right)$$

$$(z-1.005)X = X(0) - p\left(\frac{z}{z-1}\right)$$

$$X = \left(\frac{X(0)}{z-1.005}\right) - p\left(\frac{z}{(z-1)(z-1.005)}\right)$$

Using partial fractions

$$X = \left(\frac{X(0)}{z-1.005}\right) + pz\left(\left(\frac{200}{z-1}\right) - \left(\frac{200}{z-1.005}\right)\right)$$

Converting back to the time domain

$$x(k) = 1.005^k X(0) - 200p(1.005^k - 1)u(k)$$

After 10 years (k=120 payments), x(k) should be zero meaning you owe no more money

$$x(120) = 0 = \$181,939 - 200p(0.8194)$$

$$p = \$1102.05$$

Your monthly payments are \$1,103.50.

If you stretch this out to 30 years (k = 360 payments), the monthly payment becomes

$$x(360) = 0 = \$602,257 - 200p(5.0225)$$

$$p = \$599.55$$

Paying off the loan over a time span 3 times longer doesn't reduce the payments by 3 times. It's actually only 46% less. The total amount you'll pay on the loan, however, increases from \$132,246 to \$215,838.

note: That's pretty much all a business calculator is: a calculator which does z-transforms where the keys are renamed "interest rate", "initial loan value" and "number of payments."

Summary

z-transforms convert difference equations into algebraic equations in z. Whenever you're dealing with discrete events - especially when there's convonultion involved, z-transforms are a useful tool. We'll likewise be using z-transforms when analyzing discrete probability functions.

z transforms also allow you to go from the pdf to the cdf fairly easily:

$$cdf = \left(\frac{z}{z-1}\right) \cdot pdf$$