Dice Games: Farkle

Enumeration and tree analysis also apply to dice games. This looks at a popular game: Farkle.

The rules of Farkle are simple:

- Start the game by tossing six dice. You can then select any of the dice that score points to keep. If you score zero points on a toss, you say "Farkle," your turn is over, and you score zero points for that round.
- If you do score points, you may take the remaining dice and toss them. Again, if you do, you must score points on the dice you tossed (and keep) - otherwise your turn is over and you score zero points.
- If all six of your dice score points and you have zero dice left, you start over with six dice and keep going.
- Once one player reaches 10,000 points, each other player gets one more turn. After that, highest score wins.

The ways to score points are:

- Each one is worth 100 points
- Each five is worth 50 points
- Three of a kind: 100 points times the die value (three 5's is worth 500 points)
- 4 of a kind: 1000 points
- 5 of a kind: 2000 points
- 6 of a kind: 3000 points
- 1-6 Straight: 1500 points
- Three pair: 1500 points
- Four of a kind and a pair: 1500 points
- Two triplets: 2500 points

The strategy is to know when to stop tossing the dice. If you stop rolling, you keep the points you scored that round. If you elect to toss the dice, you risk scoring zero points (a Farkle) and losing everything you scored that round.

With that, you can compute several odds.

Odds of Scoring Points when Tossing One Die

This is pretty easy: there are six possible die rolls:

```
{ 1, 2, 3, 4, 5, 6 }
```

Two of the six score points. The odds are then

- 2/6 Chance of scoring points
- 4/6 Chance of a Farkle

Knowing this, you might wonder whether you should or should not toss one die. One way to analyze this is to compute the expected return.

The expected return is equal to

- (The points you expect to get if you are successful) * (the probability of success)

minus

- (the points you expect to lose if you are not successful) * (the probability of failure)

This gets a little more complicated since if you score on all six dice, you get to start over and toss six dice. Assume for now that the expected score when tossing six dice is 300 points (wild guess).

The expected return is

E(return) = (1/6) * (100 points + 300 points)	roll a 1 and then roll 6 dice
+ (1/6) * (50 points + 300 points)	roll a 5 and then roll 6 dice
- (4/6) * X	lose all X points if you Farkle
E(return) = 125 - 4/6 X	

The expected return is positive as long as X is less than 187 points (if you have less than 187 points, it's worth while to toss that last die. Otherwise, keep you points and end your turn.)

Odds of scoring points when tossing two dice:

In this case, there are 36 possibilities. A lot, but small enough for enumeration. The ones that score points are shown in red:

11	**12**	**13**	**14**	**15**	**16**
21	22	23	24	**25**	26
31	32	33	34	**35**	36
41	42	43	44	**45**	46
51	**52**	**53**	**54**	**55**	**56**
61	62	63	64	**65**	66

There are 20 ways to score points when tossing two dice. The odds of scoring are this

- 4/36 odds of both dice scoring (and you get to toss 6 dice next roll: shown in blue)
- 16/36 odds of one die scoring (shown in red)
- 16/36 odds of a Farkle

The expected return for tossing two dice is then

E(return) = (1/36) * (200 points + 300 points)	roll two 1's
+ (2/36) * (150 points + 300 points)	roll 1 & 5
+ (1/36) * (100 points + 300 points)	roll two 5's
+ (8/36) * (100 points)	roll a single 1
+ (8/36) * (50 points)	roll a single 5
- (16/36) * X	Farkle

or

E(return) = 83.333 - 16/36 * X

The expected return is positive if X < 187, meaning

- You should toss two dice if your score this round is less than 187 points.
- You should end your turn if you score this round is more than 187 points

Odds when tossing three dice

Now is when you might want to use combinatorics: there are 216 possible outcomes when tossing three dice. The odds of not rolling any 1's or 5's is:

$$p = \left(\frac{4}{6}\right)\left(\frac{4}{6}\right)\left(\frac{4}{6}\right) = 0.2963$$

The number of ways you can toss 3 dice and get no 1's or 5's is thus

$$M = 0.2963 \cdot 216 = 64$$

There are actually four more ways to score (three 2's, three 3's, etc) for

$$M = 64 + 4 = 68$$

meaning the number of ways to score when rolling three dice is

$$N - 216 - M = 148$$

This gives

- 148 / 216 the probability of scoring when tossing three dice
- 68 / 216 the probability of a Farkle

Expected Return when Tossing Three Dice:

Three of a kind: 100 * die value + 300 points (you get to roll all 6 dice next round)

```
E(return)   = (1/216) * (300 + 300)       three 1's
            + (1/216) * (200 + 300)       three 2's
            + (1/216) * (300 + 300)       three 3's
            + (1/216) * (400 + 300)       three 4's
            + (1/216) * (500 + 300)       three 5's
            + (1/216) * (600 + 300)       three 6's
```

Add to this the chance of rolling

```
            + (3/216) * (250 + 300)       115   (3 permutations)
            + (3/216) * (200 + 300)       155   (3 permutations)
            + (12/216) * (200)            11x   (12 permutations)
            + (12/216) * (150)            15x   (12 permutations)
            + (12/216) * (100)            55x   (12 permutations)
            + (48/216) * (100)            1xx   (48 permutations)
            + (48/216) * (50)             5xx   (48 permutations)

            - (84/216) * X                Farkle (84 permutations)
```

The expected return is thus

$$E(\text{return}) = 91.898 - (84/216)*X$$

The expected return is positive if X is less than 236

- You should roll three dice if you have less than 236 points in this round
- You should stop rolling the dice if you have more than 236 points

Farkle and Enumeration

When tossing all 6 dice, you can calculate the odds of each type of hand by

- Going through all 6^6 different combinations of six dice, and
- Counting how many each different combination comes up.

In Matlab, the code would consist of nested for-loops:

```
for d1=1:6
  for d2=1:6
    for d3=1:6
      for d4=1:6
        for d5=1:6
          for d6 = 1:6
            Dice = [d1,d2,d3,d4,d5,d6];
            Dice = sort(Dice);
          end
        end
      end
    end
  end
```

Once you have the value of all six dice, determine the frequency of each number

```
N = zeros(1,6);
for i=1:6
    N(i) = sum(Dice == i);
end
N = sort(N, 'descend');
```

You can then determine what type of roll it was:

```
if (N(1) == 6)
   Pair6 = Pair6 + 1;
elseif (N(1) == 5)
   Pair5 = Pair5 + 1;
elseif ((N(1)==4)*(N(2)==2))
   Pair42 = Pair42 + 1;
elseif (N(1)==4)
   Pair4 = Pair4 + 1;
elseif ((N(1)==3)*(N(2)==3))
   Pair33 = Pair33 + 1;
elseif (N(1)==3)
   Pair3 = Pair3 + 1;
elseif ((N(1)==2)*(N(2)==2)*(N(3)==2))
   Pair222 = Pair222+1;
end
```

The net result is the exact number of rolls that produce each result:

6-of-kind	5-of-kind	4+2-of-kind	4-of-kind	3+3 of kind	3 of kind	3-pairs
6	180	450	1,800	300	14,400	1,800

Can we come up with the same results using combinatorics?

6 of a kind odds (N=6)

When you start and roll 6 dice, what are the odds of rolling six of a kind?

The number of permutations is

$$M = 6^6 = 46,656$$

Enumeration works here: there are six ways to get 6 of a kind:

```
{ 111111, 222222, 333333, 444444, 555555, 666666 }
```

You can also use combinatorics. With 6-of-a-kind, your dice look like

Dice = xxxxxx

The number of combinations that result in this roll are:

- There are 6 values on a die, choose 1 for x
- There are 6 locations to place x, choose 6

$$N = \binom{6}{1}\binom{6}{6} = 6$$

5 of a kind odds (N=180)

The ways to roll 5 of a kind are

```
xxxxx y
```

where x and y are non-matching numbers. The number of permutations is

N = (6 numbers, choose one for x) (5 other numbers, pick 1 for y) (6 spots for x, pick 5)

(1 spot for y, pick 1)

$$N = \binom{6}{1}\binom{5}{1}\binom{6}{5}\binom{1}{1} = 180$$

4+2 of a kind (N = 450)

A roll of this type looks like

xxxx yy

The number of ways to get this are

- Of the 6 die values, choose 1 for x
- Of the 6 locations to place x, choose 4
- Of the remaining 5 die values, choose 1 for y
- Of the two remaining locations for y, choose 2

$$N = \binom{6}{1}\binom{6}{2} \cdot \binom{5}{1}\binom{2}{2}$$

$$N = 450$$

4 of a kind odds (N=1800)

A roll that results in 4-of-a-kind looks like:

```
xxxx yz
```

The number of permutations are

N = (6 numbers, choose 1 for x) (6 spots to place x, choose 4)

* (5 other numbers, choose 2 for y and z)(2 spots for y, choose 1) (1 spot for z, choose 1)

$$N = \binom{6}{1}\binom{5}{2}\binom{6}{4}\binom{2}{1}\binom{1}{1} = 1800$$

Two Triples (N = 300)

A hand with two triples looks like

```
xxx yyy
```

The number of combinations to do this is as follows. Since x and y have the same frequency, do these together

- Of the 6 numbers, choose two for x and y
- Of the 6 spots to place x, choose 3
- Of the 3 remaining spots to place y, choose 3

$$N = \binom{6}{2}\binom{6}{3}\binom{3}{3}$$

$$N = 300$$

Three of a Kind (N = 14,400)

A hand that counts as 3-of-a-kind looks like

```
xxx aa b
xxx a b c
```

Taking the first case

- Of the 6 dice values, choose 1 for x
- Of the 6 locations, choose 3 for x
- Of the 5 remaining dice values, choose 1 for a
- Of the 3 remaining locations, choose 2 for a
- Of the 4 remaining dice values, choose 1 for b
- Of the 1 remaining locations, choose 1 for b

$$N = \binom{6}{1}\binom{6}{3}\cdot\binom{5}{1}\binom{3}{2}\cdot\binom{4}{1}\binom{1}{1}$$

$$N = 7200$$

For the second case

- Of the 6 dice values, choose 1 for x
- Of the 6 locations, choose 3 for x
- Of the 5 remaining values, choose 3 for abc
- Of the 3 remaining spots, choose 1 for a
- Of the 2 remaining spots, choose 1 for b
- Of the 1 remaining spot, choose 1 for c

$$N = \binom{6}{1}\binom{6}{3}\cdot\binom{5}{3}\binom{3}{1}\binom{2}{1}\binom{1}{1}$$

$$N = 7200$$

Adding them together results in 14,400 combinations that result in 3-of-a-kind.

3-Pairs (N = 1800)

A roll of three pairs looks like

```
xx yy zz
```

The number of combinations that produce this is as follows. Since x, y, and z have the same frequency, do these together:

- Of the 6 die values, choose 3 for xyz
- Of the 6 spots, choose 2 for x
- Of the 4 remaining spots, choose 2 for y
- Of the 2 remaining spots, choose 2 for z

$$N = \binom{6}{3}\binom{6}{2}\binom{4}{2}\binom{2}{2}$$

$$N = 1800$$

Summary

Combinatorics allows you to calculate the number of combinations that result in different die rolls. It can be a little tricky trying to figure out how to compute this.

- You first select how many numbers you choose from those available.
- You then choose how many locations that number is placed.
- If two different values have the same frequency, evaluate these together

If done correctly, you'll end up with exact answers that match up with results you get using enumeration.