2. Combinations and Permutations: 5-Card Stud

In previous lectures, we used Monte-Carlo simulations and enumeration to compute probabilities.

- With Monte-Carlo simulations, you first play a game one time. If that works, you then repeat the experiment a large number of times. This is something that Matlab is very good at but the resulting probabilities are only approximate.
- With enumeration, you list out all possible outcomes and count how many of them were a success. Here, the results are exact. In addition, Matlab can compute probabilities using enumeration even when the total number of combinations is large (like a million), but not exceeding large.

In this lecture, we look at a third way to compute probabilities: combinatorics.

Combinatorics is the study of determining how many ways an event can happen. Assuming all events are equally likely, it also allows you to determine the probability of a certain outcome.

With combinatorics, you can calculate the total number of ways you can deal out a given hand as well as the total number of ways to get different combinations of cards (such as 4-of-a-kind, etc). The advantage of combinatorics is

- The resulting probabilities are exact (as opposed to approximate as in Monte-Carlo simulations), and
- It can take less time than enumeration

The disadvantage is it can get kind of tricky to compute some probabilities.

Definitions:

- n! "n factorial" n x (n-1) x (n-2) x ... x 2 x 1
- 0! = 1 Just define zero factorial to be one.
- p(x) "the probability of outcome x"
- ${}_nP_m$ "Permutations of n events taken m at a time". ${}_nP_m = \frac{n!}{(n-m)!}$
- ${}_nC_m$ "Combinations of n events taken m at a time" ${}_nC_m = \frac{n!}{m!\cdot(n-m)!}$
- $\binom{n}{m}$ "n choose m". Another way of writing ${}_nC_m$

Factorials:

N factorial (N!) is defined as the product of all integers from 1..N

$$N! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot N$$

N! is also the number of ways you can arrange N items.

For example, suppose 20 people are standing in a line. The total number of ways to arrange 20 people is 20!:

- There are 20 ways to pick the first person in line

- There are then 19 ways to pick the following person,
- 18 ways to pick the person after that

and so on. So, the number of ways 20 people can stand in line is

$$N = 20! = 2.43290 \cdot 10^{18}$$

The number of ways you can shuffle a deck of 52 cards is 52!

$$N = 52! = 8.0658 \cdot 10^{67}$$

With numbers of this magnitude, enumeration simply is not feasible.

Permutations: (Order Matters)

Permutations is the number of ways you can select m items from a population of size n when order matters. For example, if 20 people are trying for a volleyball team and order matters

- first person selected is outside hitter,
- next person selected is middle hitter,
- etc

then the total number of ways to select a volleyball team is

- There are 20 ways to select the fist player
- 19 ways to select the enxt player, e
- etc

or

$$M = 20 \cdot 19 \cdot 18 \cdot 17 \cdot 16 \cdot 15$$

$$M = 27,907,200$$

This can also be written as *20 Pick 6*

$$M = \left(\frac{20!}{(20-6)!}\right) = {}_{20}P_6$$

As another example, the number of ways to deal out 5 cards from a 52 card deck is 52 pick 5

$$M = {}_{52}P_5 = \left(\frac{52!}{(52-5)!}\right) = 311,875,200$$

Permutations isn't a built-in function in Matlab - but you can write your own function.

```
function [N] = nPm(n, m)

N = factorial(n) / factorial(n-m);
end
```

Matlab funciton for permutations: n pick m

If you save this as *nPm.m* you can then call it from the command window

```
>> nPm(20,6)
ans =    27907200

>> nPm(52,5)
ans =   311875200
```

Combinations (order doesn't matter)

Combinations are the number of ways you can select m items from a population of n when order does **not** matter. For example

- Select 6 players from a population of 20 for a volleyball team. The players select who plays what position.
- Draw 5 cards from a 52-card deck. You can then shuffle your hand as you like (order doesn't matter).

When order doesn't matter, permutations over-counts by m factorial (the number of ways you can arrange m items). Hence

$$ {}_nC_m = \left(\begin{array}{c} n \\ m \end{array} \right) = \left(\frac{{}_nP_m}{m!} \right) = \left(\frac{n!}{(n-m)!m!} \right) $$

This is read as *n choose m.*

For example, the number of ways you can select 6 players from a population of 20 to form a volleyball team is

$$ N = \left(\begin{array}{c} 20 \\ 6 \end{array} \right) = \left(\frac{20!}{(20-6)!6!} \right) = 38,760 $$

The number of poker hands that can be dealt (assuming order doesn't matter) is

$$ N = \left(\begin{array}{c} 52 \\ 5 \end{array} \right) = \left(\frac{52!}{(52-5)!4!} \right) = 2,598,960 $$

Matlab also doesn't have a combinations function, but it's easy to write:

```
function [N] = nCm(n, m)

N = factorial(n) / factorial(n-m);
N = N / factorial(m);

end
```

Matlab function for combinations: n choose m

Save this as nCm.m and you can call it from the command window

```
>> nCm(20,6)
ans =       38760

>> nCm(52,5)
ans =     2598960
```

Poker: 5-Card Stud

To give us something concrete to relate to, consider the poker game of 5-card stud. In this game, cards are dealt one at a time:

- Starting out, a deck of 52 cards is shuffled.
- The first card is played face down so that only the player sees this card.
- The second card is played face up so all players can see it. After two cards are played, bets are made.
- Once betting stops, a third card is played face up and betting starts over again.
- Ditto for the 4th card and 5th card.
- Once betting is finished with 5 cards for each player, the face down card is revealed and the winner is determined.

The winning hands (in order) for poker are

- Royal Flush: 10-J-Q-K-A of the same suit
- Straight-Flush: A run of 5 cards in the same suit
- 4 of a kind: Four of your cards have the same value. Ex: J-J-J-J-x
- Full-House: 3 of a kind and a pair. Ex: J-J-J-Q-Q
- Flush: All cards of the same suit
- Straight: A run of 5 cards
- 3 of a kind: Three of your cards match. Ex: J-J-J-x,y
- 2-Pair: Two pairs of cards. J-J-Q-Q-x
- Pair: Two cards match. J-J-x-y-z
- High-Card: Other. No pairs, no straights, no flush.

Suppose you want to know the probability of each type of poker hand. Three ways to determine this are

- Monte-Carlo simulations: deal out one million hands and count the frequency of each type of hand
- Enumeration: deal out every possible poker hand and count the frequency of each type of hand
- Wikipedia: look up the answer

In this lecture, we'll look at a 4th way: using combinatorics.

Poker has been analyzed to death, so the probability of each type of hand is well known. The answer you get from Wikipedia should also match the number of hands you get using enumeration (last lecture) or combinatorics (this lecture). From Wikipedia, the number of hands and odds against are as follows:

POKER HAND	COUNT	Odds Against	Payout
Royal Flush	4	649,740	800
Straight Flush	36	72,193.33	50
Four of a Kind	624	4,165	20
Full House	3,744	694.17	6
Flush	5,108	508.8	5
Straight	10,200	254.8	4
Three of a Kind	54,912	47.33	3
Two Pair	123,552	21.04	2
One Pair	1,098,240	2.37	1
High Card	1,302,536	2	0
Total:	2,598,960		

Total Number of Each Type of Poker Hand (Wikipedia)

Let's try to get these numbers using combinatorics.

Total Number of Poker Hands

From enumeration, the total number of hands you can be dealt (assuming order doesn't matter) is 2,598,960. This took 186 seconds to count in the last lecture. Using combinatorics, you can get the same result much quicker:

$$N = \binom{52}{5} = 2,598,960$$

Probability of getting a Royal Flush

Once you know the total number of poker hands possible, you can compute the probability of any given hand as

$$p(x) = \frac{\text{the total number of hands that are x}}{2,598,960}$$

The easiest is a royal flush: you can use enumeration to list out all possible royal flushes.

- 10-J-Q-K-A in spades
- 10-J-Q-K-A in hearts
- 10-J-Q-K-A in diamonds
- 10-J-Q-K-A in clubs

Hence, the probability of a royal flush is

$$p(\text{royal flush}) = \left(\frac{4}{2,598,960}\right) = \frac{1}{649,740}$$

The odds against getting a royal flush are 649,740 : 1

Another way to think of this, if you played one million poker games, you should on average get 1.53 royal flushes.

4 of a kind:

From enumeration, there are 624 ways to get 4-of-a-kind. You can also get this using combinatorics.

A hand that is 4-of-a-kind looks like

- hand = x-x-x-x-y

where all x's have the same value (Ace through King) and all y's have a different value (something different from x). The number of ways to get such a hand is:

xxxx:

- There are 13 values in a deck. Choose one of these for x.
- There are 4 x's in the deck. Of those four, choose four

y:

- There are 12 non-x values to choose from for y. Of those 12, choose 1.
- Of the 4 y's in the deck, choose one

Meaning

$$M = \binom{13}{1}\binom{4}{4}\cdot\binom{12}{1}\binom{4}{1}$$

$$M = 624$$

Another equally valid way to think of this is

xxxx:

- There are 13 values in a deck. Choose one of these for x.
- There are 4 x's in the deck. Of those four, choose four

y:

- There are 48 remaining cards in the deck, choose one for y

$$M = \binom{13}{1}\binom{4}{4}\cdot\binom{48}{1}$$

$$M = 624$$

Full-House: M = 3744

From enumeration, there are 3744 ways to get a full-house. Using combinatorics, try to get the same result.

A full-house looks like

hand = xxx yy

where x and y are different values. The number of ways to get a full-house are:

xxx:

- There are 13 values to choose for x. Choose one.
- Of the 4 cards with that value, choose three

yy:

- There are 12 remaining values to choose for y. Choose 1
- Of the 4 cards of that value, choose two

Meaning

$$M = \binom{13}{1}\binom{4}{3} \cdot \binom{12}{1}\binom{4}{2}$$

$$M = 3744$$

Flush: M = 5108

A flush consists of five cards of the same suit. A hand that contains a flush looks like

Suit = aaaaa

where 'a' is clubs, diamonds, hearts, or spades. The total number of ways to get a flush are:

- Of the four suits, choose one (4 choose 1)
- Of the 13 cards in that suit, choose five (13 choose 5)

$$M = \binom{4}{1}\binom{13}{5} = 5,148$$

This includes straight-flushes (N = 40). Removing these gives 5,108 different flushes

Straight: M = 10,200

- A straight can start with an Ace (Ace-5) through a 10 (10-J-Q-K-A). That gives 40 starting cards (10 values in 4 suits)
- Of the four cards of the next value, choose one (4 choose 1)
- Of the four cards of the next value, choose one (4 choose 1)

so

$$M = 40 \cdot \binom{4}{1}\binom{4}{1}\binom{4}{1}\binom{4}{1} = 10,240 \quad \textit{all straignts}$$

This also counts straight-flushes (M=40). Removing these gives the correct answer:

$$M = 10,200$$

3-of-a-kind (M = 54,912)

A hand that's 3-of-a-kind looks like

hand = xxx ab

where x, a, and b are different values. An incorrect way to compute the number of hands that result in 3-of-a-kind

xxx:

- Of the 13 values from Ace to King, choose one for x
- Of the 4 x's in the deck, choose 3

a:

- Of the remaining 12 values, choose one for a
- Of the 4 a's in the deck, choose 1

b:

- Of the remaining 11 values, choose one for b
- Of the 4 b's in the deck, choose 1

Meaning

$$M = \binom{13}{1}\binom{4}{3} \cdot \binom{12}{1}\binom{4}{1} \cdot \binom{11}{1}\binom{4}{1}$$

$$M = 109,824$$

This answer is off by 2x due to double-counting ab: this computation treats ab=45 as distinct from ab=54.

When the frequency of terms are the same, you need to select these together. A correct way to compute M is

xxx:

- Of the 13 values from Ace to King, choose one for x
- Of the 4 x's in the deck, choose 3

ab:

- Of the remaining 12 values, choose two for ab
- Of the 4 a's in the deck, choose 1
- Of the 4 b's in the deck, choose 1

Meaning

$$M = \binom{13}{1}\binom{4}{3} \cdot \binom{12}{2}\binom{4}{1}\binom{4}{1}$$

$$M = 54,912$$

2-Pair: M = 123,552

A hand that contains 2-pair looks like

hand = xx yy a

To compute the number of possible hands using combinatorics, note that x and y have the same frequency. Compute these together

xx yy

- Of the 13 values, choose two for x and y
- Of the 4 x's in the deck, choose 2
- Of the 4 y's in the deck, choose 2

a

- Of the 44 cards that are not x or y, choose 1

Meaning

$$M = \binom{13}{2}\binom{4}{2}\binom{4}{2} \cdot \binom{44}{1}$$

$$M = 123,552$$

Pair: M = 1,098,240

A hand that contains a pair looks like

hand = xx abc

To compute the number of hands that look like this using combinatorics, group 'abc' together since they have the same frequency.

xx:

- Of the 13 values, choose one for x
- Of the 4 x's in the deck, choose 2

abc

- Of the remaining 12 values, choose three for 'abc'
- Of the four a's, choose one
- Of the four b's, choose one
- Of the four c's, choose one

meaning

$$M = \binom{13}{1}\binom{4}{2} \cdot \binom{12}{3}\binom{4}{1}\binom{4}{1}\binom{4}{1}$$

$$M = 1,098,240$$

High-Card: M = 1,302,540

A hand that's a high-card looks like

hand = abcde

where the hand is not a straight (10,200) or a flush (5,108), a straight-flush (36), or a royal flush (4)

To compute this using combinatorics, note that all five cards have the same frequency, so analyze these together:

abcde:

- Of the 13 values, choose five for abcde
- Of the 4 a's, choose one
- Of the 4 b's, choose one
- Of the 4 c's, choose one
- Of the 4 d's, choose one
- Of the 4 e's, choose one

$$M = \binom{13}{5}\binom{4}{1}\binom{4}{1}\binom{4}{1}\binom{4}{1}\binom{4}{1}$$

$$M = 1,317,888$$

This includes straignts, flushes, straight-flushes, and royal flushes. Removing these gives

$$M = 1,302,540$$

Expected Return:

In video poker, the payout for each type of hand is as follows (from Wikipedia)

POKER HAND	COUNT	Odds Against	Payout
Royal Flush	4	649,740	800
Straight Flush	36	72,193.33	50
Four of a Kind	624	4,165	20
Full House	3,744	694.17	6
Flush	5,108	508.8	5
Straight	10,200	254.8	4
Three of a Kind	54,912	47.33	3
Two Pair	123,552	21.04	2
One Pair	1,098,240	2.37	1
High Card	1,302,536	2	0
Total:	2,598,960		

What this means is as follows:

- If you bet $1 and the payout is zero, you lose your $1 bet.
- If the payout is N, you get $N back.

The expected return for playing a hand of poker is then the sum of

- The probability of each hand,
- Times the net payout (1 + N)

Calculating this:

POKER HAND	COUNT	Odds Against	Payout	Return
Royal Flush	4	649,740	800	$0.001
Straight Flush	36	72,193.33	50	$0.001
Four of a Kind	624	4,165	20	$0.005
Full House	3,744	694.17	6	$0.009
Flush	5,108	508.8	5	$0.010
Straight	10,200	254.8	4	$0.016
Three of a Kind	54,912	47.33	3	$0.063
Two Pair	123,552	21.04	2	$0.095
One Pair	1,098,240	2.37	1	$0.423
High Card	1,302,536	2	0	$0.000
Total:	2,598,960			$0.622

The net return on your $1 bet is $0.622: for every dollar you bet, you expect to lose 38 cents. This is a pretty lousy game of slots.

Summary

Previous lectures used Monte-Carlo simulations and enumeration to compute the probability of events. With combinatorics, you can get the same exact answers you get using enumeration. It can be a little tricky, but you get the same result with a lot less number crunching.