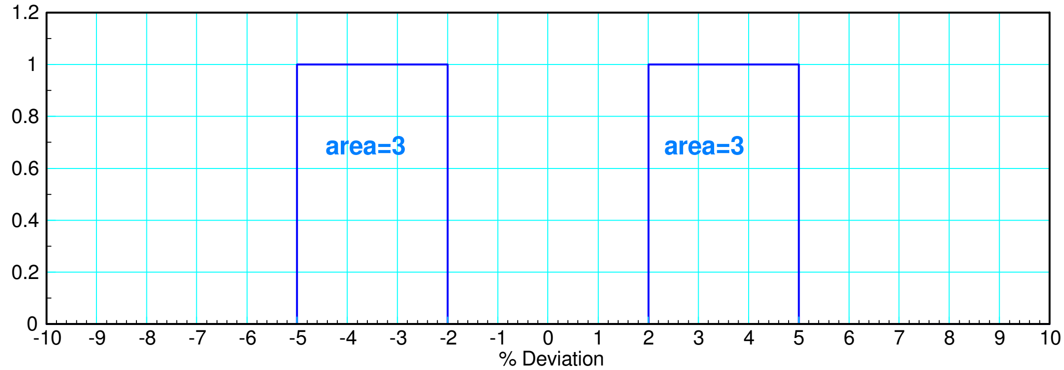
ECE 341 - Test #2

Continuous Probability - Summer 2024

1) Continuous PDF

A 5% tolerance rersistor often has a pdf as shown below (resistors which are within 2% of rated value are removed and sold as 2% or 1% resistors).



a) Determine a scalar so that this is a valid pdf (i.e. the total area = 1.0000)

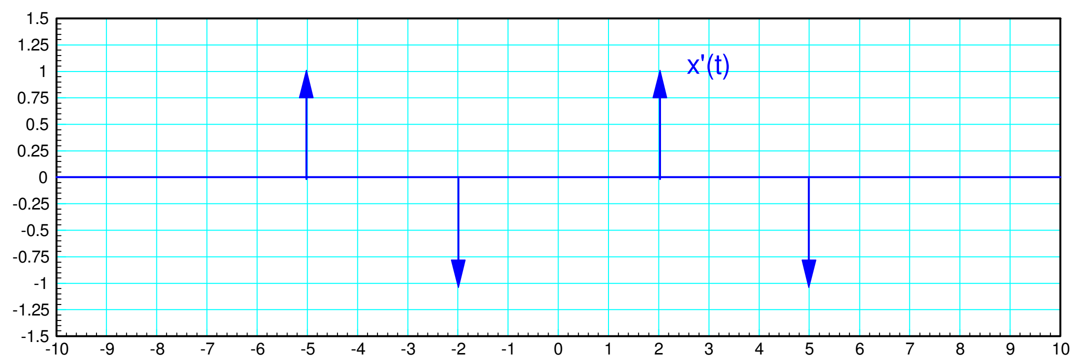
The area is 6

k = 1/6

b) Determine the moment generating function (i.e. LaPlace transform)

Differentiate to get delta functions. Once you have delta functions, the LaPlace transform is easy:

$$6x'(s) = e^{5s} - e^{2s} + e^{-2s} - e^{-5s}$$



Integrate to get back to the pdf

$$X(s) = \left(\frac{1}{6s}\right)\left(e^{5s} - e^{2s} + e^{-2s} - e^{-5s}\right)$$

2) Uniform Distribuitions

Let A, B, and C be continuous uniform distributions

- A = uniform over the interval of (3, 7)
- B = uniform over the interval of (1, 2),
- X = A + B

Use moment generating functions to determine the pdf for X (i.e. LaPlace Transforms)

$$A(s) = \left(\frac{1}{4s}\right)(e^{-3s} - e^{-7s})$$

$$B(s) = \left(\frac{1}{s}\right)(e^{-s} - e^{-2s})$$

$$X(s) = A(s) \cdot B(s)$$

$$X(s) = \left(\frac{1}{4s}\right)(e^{-3s} - e^{-7s}) \cdot \left(\frac{1}{s}\right)(e^{-s} - e^{-2s})$$

Multiply out

$$X(s) = \left(\frac{1}{4s^2}\right)(e^{-4s} - e^{-5s} - e^{-8s} + e^{-9s})$$

Take the inverse LaPlace transform

$$x(k) = \left(\frac{1}{4}\right)((t-4)u(t-4) - (t-5)u(t-5) - (t-8)u(t-8) + (t-9)u(t-9))$$

3) Gamma CDF

Let A, B be continuous exponential distributions:

- A has a mean of 2 seconds $\quad a(t) = \frac{1}{2}e^{-t/2}u(t) \quad A(s) = \left(\frac{1/2}{s+1/2}\right)$
- B has a mean of 5 seconds $\quad b(t) = \frac{1}{5}e^{-t/5}u(t) \quad B(s) = \left(\frac{1/5}{s+1/5}\right)$

The moment generating function for the cdf of Y = two A's and one B happens is

$$Y(s) = \left(\frac{1}{s}\right)\left(\frac{1/2}{s+1/2}\right)^2\left(\frac{1/5}{s+1/5}\right)$$

Determine the equation for the cdf (i.e. take the inverse LaPlace transform)

Do a partial fraction expansion

$$Y(s) = \left(\frac{a}{s}\right) + \left(\frac{b}{(s+1/2)^2}\right) + \left(\frac{c}{s+1/2}\right) + \left(\frac{d}{s+1/5}\right)$$

{a, b, d} can be found using the cover-up method

$$a = \left(\left(\frac{1/2}{s+1/2}\right)^2\left(\frac{1/5}{s+1/5}\right)\right)_{s=0} = 1$$

$$b = \left(\left(\frac{1}{s}\right)\left(\frac{1/5}{s+1/5}\right)\right)_{s=-1/2} = \frac{1}{3}$$

$$d = \left(\left(\frac{1}{s}\right)\left(\frac{1/2}{s+1/2}\right)^2\right)_{s=-1/5} = 2\frac{7}{9}$$

Find c by placing all terms over a common denominator. The numerator is then...

$$\frac{1}{20} = a\left(s+\frac{1}{2}\right)^2\left(s+\frac{1}{5}\right) + b(s)\left(s+\frac{1}{5}\right) + c(s)\left(s+\frac{1}{2}\right)\left(s+\frac{1}{5}\right) + d(s)\left(s+\frac{1}{2}\right)^2$$

Matching the s3 terms

$$0 = a + c + d$$

$$c = 1\frac{7}{9}$$

So

$$Y(s) = \left(\frac{1}{s}\right) + \left(\frac{1/3}{(s+1/2)^2}\right) + \left(\frac{1.7778}{s+1/2}\right) + \left(\frac{-2.7778}{s+1/5}\right)$$

and

$$y(t) = (1 + 0.333te^{-t/2} + 1.7778e^{-t/2} - 2.7778e^{-t/5})u(t)$$

4) Central Limit Theorem

The Dungeons and Dragons spell *Meteor Swarm* does 20-120 damage (the sum of twenty 6-sided dice)

y = 20d6

Use a normal approximation to determine the probability that the total damage is more than 99.5

die	d4	d6	d8	d10	d12
mean	2.5000	3.5000	4.5000	5.5000	6.5000
variance	1.2500	2.9167	5.2500	8.2500	11.9167

mean of y	standard deviation of y	z-score for sum = 99.5	p(sum > 99.5)
70	**7.6388**	**3.8619**	**0.000 056 13** 56 in one million

Mean: Scales with the number of dice

$$\mu_y = 20 \cdot \mu_{d6}$$

$$\mu_y = 20 \cdot 3.5 = 70$$

Standard Deviation: The variance scales with the number of dice

$$\sigma_y^2 = 20 \cdot \sigma_{d6}^2$$

$$\sigma_y^2 = 20 \cdot 2.9176 = 58.3520$$

$$\sigma_y = \sqrt{58.352} = 7.6388$$

z-Score: The distance of 99.5 from the mean

$$z = \left(\frac{99.5 - 70}{7.6388}\right) = 3.8619$$

From a standard normal table, this z-score corresponds to a probability of 0.0005613

5) Testing with Normal pdf

Two wizards in Dungeons and Dragons cast spells. Let

- A be the damage done by a *Flame Strike* spell (the sum of eight 6-sided dice: 8d6)
- B be the damage done by a *Firestorm* spell (sum of seven 10-sided dice: 7d10)

die	d4	d6	d8	d10	d12
mean	2.5000	3.5000	4.5000	5.5000	6.5000
variance	1.2500	2.9167	5.2500	8.2500	11.9167

Use a normal approximation to determine the probability that A > B

	A = 8d6	B = 7d10	W = A - B
mean	**28**	**38.5**	**-10.5**
variance	**23.3333**	**57.75**	**81.08**
z-Score	**1.1666**		
p(A > B)	**0.12302**		

Mean:

$$\mu_a = 8 \cdot \mu_{d6} = 28$$

$$\mu_b = 7 \cdot \mu_{d10} = 38.5$$

$$\mu_w = \mu_a - \mu_b = -10.5$$

Variance

$$\sigma_a^2 = 8 \cdot \sigma_{d6}^2 = 23.333$$

$$\sigma_b^2 = 7 \cdot \sigma_{d10}^2 = 57.75$$

$$\sigma_w^2 = \sigma_a^2 + \sigma_b^2 = 81.08$$

$$\sigma_w = \sqrt{81.08}$$

z-score

$$z = \left(\frac{\mu_w - 0}{\sigma_w}\right)$$

$$z = \left(\frac{-10.5}{9.0044}\right) = 1.1661$$

Use a standard normal table to convert this z-score to an area

area of tail = 0.12302