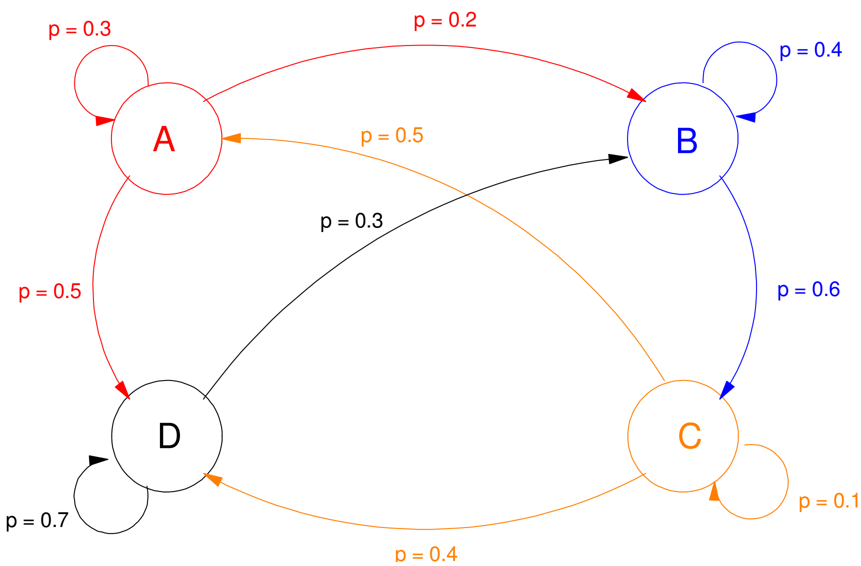
ECE 341 - Homework #11

Markov Chains. Summer 2024

Markov Chains

Four people are playing a game of hot potato. Each second, a player can keep the potato or pass it to another player. The probability of each decision are as follows:



1) Assume player A starts with the potato. Determine the probability that each player has the potato after 10 tosses using matrix multiplication.

$$\begin{bmatrix} A(k+1) \\ B(k+1) \\ C(k+1) \\ D(k+1) \end{bmatrix} = \begin{bmatrix} 0.3 & 0 & 0.5 & 0 \\ 0.2 & 0.4 & 0 & 0.3 \\ 0 & 0.6 & 0.1 & 0 \\ 0.5 & 0 & 0.4 & 0.7 \end{bmatrix} \begin{bmatrix} A(k) \\ B(k) \\ C(k) \\ D(k) \end{bmatrix}$$

```
>> M = [0.3,0,0.5,0;0.2,0.4,0,0.3;0,0.6,0.1,0;0.5,0,0.4,0.7]

    0.3000         0    0.5000         0
    0.2000    0.4000         0    0.3000
         0    0.6000    0.1000         0
    0.5000         0    0.4000    0.7000

>> X0 = [1;0;0;0];
>> M^10 * X0

a    0.1245
b    0.2614
c    0.1744
d    0.4397
```

After ten tosses, there is a 12.45% chance that A has the ball...

2) Assume player A starts with the potato. Determine the probability that player A has the potato after k tosses using z-transforms.

- What is the probability that player A has the potato after infinite tosses?

```
>> A = M;
>> X0 = [1;0;0;0];
>> C = [1,0,0,0];
>> D = 0;
>> G = ss(A,X0,C,D,1);
>> zpk(G)

   (z-0.8875) (z^2 - 0.3125z + 0.1127)
------------------------------------------
(z-1) (z-0.1344) (z^2 - 0.3656z + 0.2009)

Sampling time (seconds): 1
```

Multiply by z to get the z-transform for A(k)

$$A(z) = \left(\frac{(z-0.8875)(z^2-0.3125z+0.1127)z}{(z-1)(z-0.1344)(z^2-0.3656z+0.2009)} \right)$$

Do a patrial fraction expansion

$$A(z) = \left(\left(\frac{0.1245}{z-1} \right) + \left(\frac{0.4547}{z-0.1344} \right) + \left(\frac{0.2153\angle -12.28^0}{z-0.4482\angle 65.93^0} \right) + \left(\frac{0.2153\angle 12.28^0}{z-0.4482\angle -65.93^0} \right) \right) z$$

Take the inverse z-transform

$$a(k) = 0.1245 + 0.4547(0.1344)^k + 0.4306(0.4482)^k \cos\left(65.93^0 \cdot k + 12.28^0\right) \quad \text{k>=0}$$

$$a(\infty) = 0.1245$$

3) Assume player A starts with the potato. Determine the probability that player A has the potato after k tosses using eigenvalues and eigenvectors.

```
>> [M,V] = eig(A)

   0.2244            0.6084                0.6084                -0.8078
   0.4713           -0.3593 - 0.0286i   -0.3593 + 0.0286i     0.0153
   0.3142           -0.1426 + 0.4979i   -0.1426 - 0.4979i     0.2675
   0.7930           -0.1065 - 0.4694i   -0.1065 + 0.4694i     0.5250

   1.0000            0.1828 - 0.4092i    0.1828 + 0.4092i     0.1344

>> X0 = [1;0;0;0];
>> IC = inv(M)*X0

   0.5547 - 0.0000i
   0.3458 - 0.0753i
   0.3458 + 0.0753i
  -0.5629 + 0.0000i
```

Translation...

$$a(k) = 0.5547\begin{bmatrix} 0.2244 \\ 0.4713 \\ 0.3142 \\ 0.7930 \end{bmatrix}(1)^k$$

$$+(0.3458 - j0.0753)\begin{bmatrix} 0.6084 \\ -0.3593 - j0.0286 \\ -0.1426 + j0.4979 \\ -0.1065 - j0.4694 \end{bmatrix}(0.1828 - j0.4092)^k$$

$$+(0.3458 + j0.0753)\begin{bmatrix} 0.6084 \\ -0.3593 + j0.0286 \\ -0.1426 - j0.4979 \\ -0.1065 + j0.4694 \end{bmatrix}(0.1828 + j0.4092)^k$$

$$+(-0.5620)\begin{bmatrix} -0.8078 \\ 0.0153 \\ 0.2675 \\ 0.5250 \end{bmatrix}(0.1344)^k$$

Markov Chains with Absorbing States

Problem 4 & 5: Two teams, A and B, are playing a match made up of N games. For each game

- Team A has a 40% chance of winning
- There is a 25% chance of a tie, and
- Team B has a 35% chance of winning

In order to win the match, a team must be up by 2 games.

4) Determine the probabilty that team A wins the match after k games for k = {0 ... 10} using matrix multiplication.

This is a Markov chain

$$\begin{bmatrix} P2(k+1) \\ P1(k+1) \\ P0(k+1) \\ M1(k+1) \\ M2(k+1) \end{bmatrix} = \begin{bmatrix} 1 & 0.4 & 0 & 0 & 0 \\ 0 & 0.25 & 0.4 & 0 & 0 \\ 0 & 0.35 & 0.25 & 0.4 & 0 \\ 0 & 0 & 0.35 & 0.25 & 0 \\ 0 & 0 & 0 & 0.35 & 1 \end{bmatrix} \begin{bmatrix} P2(k) \\ P1(k) \\ P0(k) \\ M1(k) \\ M2(k) \end{bmatrix}$$

In Matlab

```
>> A =
[1,0.4,0,0,0;0,0.25,0.4,0,0;0,0.35,0.25,0.4,0;0,0,0.35,0.25,0;0,0,0,0.35,1]

    1.0000    0.4000         0         0         0
         0    0.2500    0.4000         0         0
         0    0.3500    0.2500    0.4000         0
         0         0    0.3500    0.2500         0
         0         0         0    0.3500    1.0000

>> X0 = [0;0;1;0;0];
>> C = [1,0,0,0,0];
>> for n=0:10
   y = C * A^n * X0;
   disp([n,y]);
   end

      n      p(A wins)
      0      0
      1      0
    2.0000    0.1600
    3.0000    0.2400
    4.0000    0.3148
    5.0000    0.3696
    6.0000    0.4133
    7.0000    0.4470
    8.0000    0.4734
    9.0000    0.4939
   10.0000    0.5099
```

5) Determine the z-transform for the probability that A wins the match after k games

- From the z transforms, determine the explicit function for p(A) wins after game k.

```
>> G = ss(A, X0, C, 0, 1);
>> zpk(G)

Zero/pole/gain:
           0.16 (z-0.25)
------------------------------------
(z-1) (z-0.7792) (z-0.25) (z+0.2792)

Sampling time (seconds): 1
```

meaning

$$A(z) = \left(\frac{0.16(z-0.25)z}{(z-1)(z-0.7792)(z-0.25)(z+0.2792)} \right)$$

Do a partial fraction expansion

$$A(z) = \left(\left(\frac{0.5664}{z-1} \right) + \left(\frac{-0.6846}{z-0.7792} \right) + \left(\frac{0}{z-0.25} \right) + \left(\frac{0.1182}{z+0.2792} \right) \right) z$$

Take the inverse z-transform

$$a(k) = \left(0.5664 - 0.6846(0.7792)^k + 0.1182(-0.2792)^k \right) u(k)$$

Problem 6: Two players are playing a game of tennis. To win a game, a player must win 4 points *and* be up by 2 points.

- If player A reaches 4 points and player B has less than 3 points, the game is over and player A wins.
- If player A reaches 4 points and player B has 3 points, then the game reverts to 'win by 2' rules. Both players keep playing until one of them is up by 2 games.

Suppose:

- Player A has a 65% chance of winning any given point
- Player B has a 35% chance of winning any given point.

What is the probabilty that player A wins the game (first to 4 games, win by 2)?

- Note: This is a combination of a binomial distribution (A has 4 points while B has 0, 1, or 2 points) along with a Markov chain (A and B both have 3 points, at which point it becomes a win-by-2 series)

This is a conditional probability. The ways A can win are:

- A wins when the match is 3-0 binomial distribution
- A wins when the match is 3-1 binomial distribution
- A wins when the match is 3-2 binomial distribution
- A wins a tie-breaker after going 3-3 Markov chain

From a binomial distribution, the odds are

3-0:

$$p = \binom{3}{3}(0.65)^3(0.35)^0 = 0.2746$$

3-1:

$$p = \binom{4}{3}(0.65)^3(0.35)^1 = 0.3845$$

3-2:

$$p = \binom{5}{3}(0.65)^3(0.35)^2 = 0.3364$$

3-3:

$$p = \binom{6}{3}(0.65)^3(0.35)^3 = 0.2355$$

If you are at 3-3, it's now a Markov chain where you have to win by two:

$$\begin{bmatrix} P2(k+1) \\ P1(k+1) \\ P0(k+1) \\ M1(k+1) \\ M2(k+1) \end{bmatrix} = \begin{bmatrix} 1 & 0.65 & 0 & 0 & 0 \\ 0 & 0 & 0.65 & 0 & 0 \\ 0 & 0.35 & 0 & 0.65 & 0 \\ 0 & 0 & 0.35 & 0 & 0 \\ 0 & 0 & 0 & 0.35 & 1 \end{bmatrix} \begin{bmatrix} P2(k) \\ P1(k) \\ P0(k) \\ M1(k) \\ M2(k) \end{bmatrix}$$

The probability that A wins this series is:

```
>> A = [1,0.65,0,0,0;0,0,0.65,0,0;0,0.35,0,0.65,0;0,0,0.35,0,0;0,0,0,0.35,1]

    1.0000    0.6500         0         0         0
         0         0    0.6500         0         0
         0    0.3500         0    0.6500         0
         0         0    0.3500         0         0
         0         0         0    0.3500    1.0000

>> X0 = [0;0;1;0;0];
>> A^100 * X0

    0.7752
         0
    0.0000
         0
    0.2248
```

The probability A wins starting from 3-3 is 0.7752

The total odds of A winning are then

$$p(A) = p \cdot p(3-0) + p \cdot p(3-1) + p \cdot p(3-2) + p(3-3)p(A|3-3)$$

$$p(A) = (0.65)(0.2746) + (0.65)(0.3845) + (0.65)(0.3364) + (0.2355)(0.7752)$$

$$p = 0.8296$$

A player who has a 65% chance of winning any given point has an 82.96% chance of winning a tennis match