ECE 341 - Homework #8

Gamma, Poisson, & Normal Distributions. Summer 2024

Gamma Distributions

Let A be an exponential distribution with a mean of 30 seconds

The time until the next customer arrives

Let B be the time until three customers arrive (B has a gamma distribution)

1) Determine the pdf for B using LaPlace transforms.

- From your results, determine the pdf at B=20

$$A = \left(\frac{1/30}{s+1/30}\right)$$

$$B = \left(\frac{1/30}{s+1/30}\right)^3$$

Use a table of LaPlace transforms

$$t^2 e^{-bt} u(t) \rightarrow \left(\frac{2}{(s+b)^3}\right)$$

$$B = \left(\frac{(1/30)^3}{2}\right)\left(\frac{2}{(s+1/30)^3}\right)$$

$$b(t) = \left(\frac{1}{2(30)^3}\right) t^2 e^{-t/30} u(t)$$

At t = 20

$$b(20) = 0.003803$$

```
>> 1/(2*30*30*30) * t^2 * exp(-t/30)

ans =   0.003803089770612
```

2) Determine the pdf of B using convolution

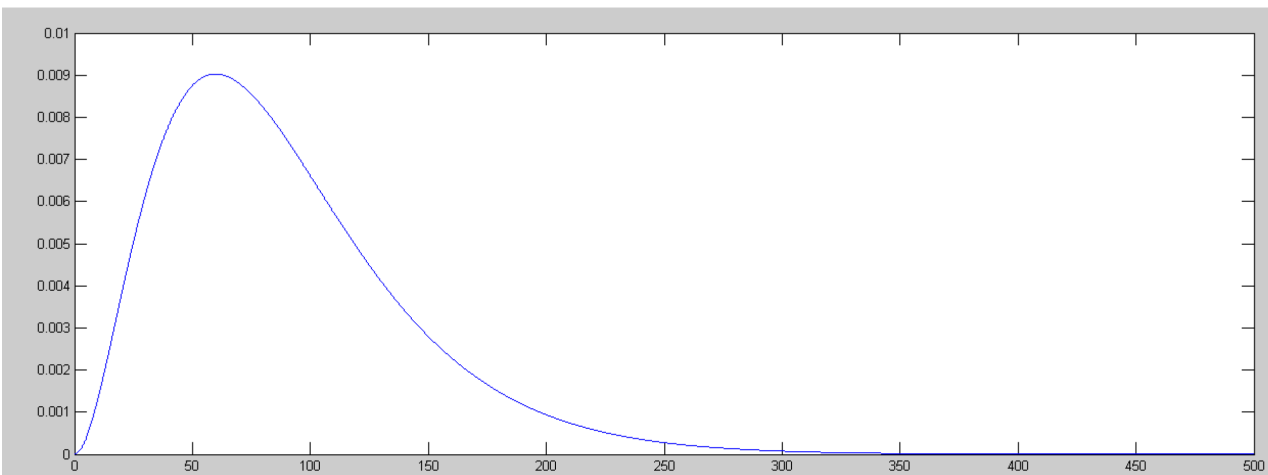
- From your results, determine the pdf at B = 20

```
>> dt = 0.01;
>> t = [0:dt:500]';

>> A = 1/30 * exp(-t/30);
>> A2 = conv(A, A) * dt;
>> B = conv(A2,A) * dt;

>> B = B(1:length(t));

>> t(20/dt + 1)

ans =    20

>> B(20/dt + 1)
```

ans =   0.003808796306813

exact answer

```
ans =   0.003803089770612
```



Poisson Distributions

3) Determine the probability that 2 customers will arrive within 60 seconds (0 < t < 60)

- Let A be an exponential distribution with a mean of 30 seconds
  The time until the next customer arrives

The probability that two customers arrive in 60 seconds:

Using moment generating functions

$$pdf = \left(\frac{1/30}{s+1/30}\right)^2$$

$$cdf = \left(\frac{1/30}{s+1/30}\right)^2\left(\frac{1}{s}\right)$$

$$cdf = \left(\frac{1}{s}\right) + \left(\frac{A}{(s+1/30)^2}\right) + \left(\frac{B}{s+1/30}\right)$$

$$\left(\frac{1}{30}\right)^2 = (s+1/30)^2 + A\cdot(s) + B\cdot(s)(s+1/30)$$

$$\left(\frac{1}{30}\right)^2 = s^2 + \left(\frac{2}{30}\right)s + \left(\frac{1}{30}\right)^2 + As + B\left(s^2 + \left(\frac{1}{30}\right)s\right)$$

B = -1

A = -1/30

$$cdf = \left(\frac{1}{s}\right) + \left(\frac{-1/30}{(s+1/30)^2}\right) + \left(\frac{-1}{s+1/30}\right)$$

$$cdf = \left(1 - \left(\frac{1}{30}\right)te^{-t/30} - e^{-t/30}\right)u(t)$$

At t = 60

```
>> t = 60;
>> 1 - 1/30*t*exp(-t/30) - exp(-t/30)

ans =   0.593994150290162
```

Using convolution

```
>> dt = 0.01;
>> t = [0:dt:500]';
>> A = 1/30 * exp(-t/30);
>> A2 = conv(A, A) * dt;
>> A2 = A2(1:length(t));
>> plot(t,A2)

>> sum(A2(1:6001)) * dt

ans =

    0.594327536187983
```

4) In D&D, you do double damage (critical hit) if you roll a natural 20 on a 20-sided die (p = 5%).

Using a binomial pdf, determine the probability of rolling a natural 20 three times in thirty rolls

$$p(m) = \binom{n}{m} p^m (1-p)^{n-m}$$

$$p(3) = \binom{30}{3} \left(\tfrac{1}{20}\right)^3 \left(\tfrac{19}{20}\right)^{27}$$

$$p(10) = 0.1270$$

Using a Poisson approximation, determine the probability rolling a natrual 20 three times in thirty rolls.

$$\lambda = np = (30)\left(\tfrac{1}{20}\right) = 1.5$$

$$f(x) = \tfrac{1}{x!} \cdot \lambda^x \cdot e^{-\lambda}$$

$$f(x=3) = \tfrac{1}{3!} \cdot (1.5)^3 \cdot e^{-1.5}$$

$$f(x=3) = 0.12551$$

Normal Distribution

- Let x be a random number from a normal distribution with a mean of 15 and a standard deviation of 5
- Let y be a random number from a normal distribution with a mean of 10 and a standard deviation of 6
- Let z be a random number from a normal distribution with a mean of 5 and a standard deviation of 7

5) Let F = x + y. Determine the probability that F > 30

a) Using a z-score

$$\mu_f = \mu_x + \mu_y = 25$$

$$\sigma_f^2 = \sigma_x^2 + \sigma_y^2 = 5^2 + 6^2 + 61$$

$$\sigma_f = \sqrt{61} = 7.801$$

The z-score is

$$z = \left(\frac{30-\mu}{\sigma}\right) = \left(\frac{30-25}{7.801}\right) = 0.6402$$

From a normal table (or StatTrek), this corresponds to a probability of 0.26102

There is a 26.10% chance the sum will be more than 30

b) Using a Monte-Carlo simulation with 100,000 samples of F

Matlab Code:

```
Wins = 0;

for i = 1:1e5
    x = randn*5 + 15;
    y = randn*6 + 10;
    f = x + y;
    if(f > 30)
        Wins = Wins + 1;
    end
end
p = Wins / 1e5


p =    0.2614
p =    0.2617
p =    0.2611
```

From a Monte Carlo simulation, p = 26.1%

6) Let G = x + y + z. Determine the probability that G > 40

a) Using a z-score

$$\mu_g = \mu_x + \mu_y + \mu_z = 30$$

$$\sigma_g^2 = \sigma_x^2 + \sigma_y^2 + \sigma_z^2$$

$$\sigma_g^2 = 5^2 + 6^2 + 7^2 = 110$$

$$\sigma_g = \sqrt{110} = 10.4881$$

The z-score for G>40 is

$$z = \left(\frac{40-\mu}{\sigma}\right) = 0.9534$$

From StatTrek, this corresponds to a probability of 0.17019

b) Using a Monte-Carlo simulation with 100,000 samples of G

Code:

```
Wins = 0;

for i = 1:1e5
    x = randn*5 + 15;
    y = randn*6 + 10;
    z = randn*7 + 5;
    g = x + y + z;
    if(g > 40)
        Wins = Wins + 1;
    end
end
p = Wins / 1e5

p =    0.1712
p =    0.1693
p =    0.1712
```

There is abou ta 17% chance the total will be more than 40