ECE 341 - Homework #2

Combinatorics & Card Games. Summer 2024

Combinatorics in Bridge

The card game *bridge* uses a 52-card deck. Each person is dealt 13 cards for their hand.

1a) How many different hands are possible? (order doesn't matter)

$$N = \binom{52}{13} = 635,013,559,600$$

1b) What is the probability of having 6 cards of one suit in your hand?

M = (4 suits choose 1)(13 cards choose 6)(39 remaining cards choose 7)

$$M = \binom{4}{1}\binom{13}{6}\binom{39}{7} = 105,574,751,568$$

$$p = \frac{M}{N} = \frac{\binom{4}{1}\binom{13}{6}\binom{39}{7}}{\binom{52}{13}} = 0.166256$$

There is a 16.6% chance of having six cards of one suit

(note: this includes the chance of two suits having 6 cards or a 6-card suit and a 7-card suit (the remainingn 7 cards can be anything)

2) What is the probability of having six face-cards (Jacks, Queens, Kings, or Aces)?

There are 16 face cards in the deck. The number of combinations that include six face-cards is

M = (16 face cards choose 6)(36 other chards choose 7)

$$M = \binom{16}{6}\binom{36}{7} = 66,848,221,440$$

The odds are then

$$p = \frac{M}{N} = \frac{\binom{16}{6}\binom{36}{7}}{\binom{52}{13}} = 0.105271$$

There is a 10.52% chance of being dealt six face-cards

3) Check your answer using Matlab and a Monte-Carlo simulation with

- A 52-card deck
- 13-card hands, and
- 100,000 hands

Results:

```
     6 of suit    6 face cards
        16388       10628

Elapsed time is 7.991061 seconds.
>>
```

	Calculated	Monte-Carlo
6 of a suit	16,625	16,388
6 face cards	10,527	10,628

Expected Results with 100,000 Hands

Matlab Code

```
% Bridge
tic
Face = 0;
Suit6 = 0;

for i0 = 1:1e5

   X = rand(1,52);
   [a,Deck] = sort(X);
   Hand = Deck(1:13);
   Value = mod(Hand-1,13) + 1;
   Suit = ceil(Hand/13);

% check for number of face cards7 of a suit
   N = zeros(1,4);
   for n=1:4
      N(n) = sum(Suit == n);
      end

   N = sort(N, 'descend');

   if(N(1) == 6)
      Suit6 = Suit6 + 1;
      end

% check for high-cards
   N = sum(Value == 1) + sum(Value > 10);
   if(N == 6)
      Face = Face + 1;
      end

   end

clc
disp('    6 of suit    6 face cards')
disp([Suit6, Face])
toc
```

In 6-card poker, you're dealt 6 cards

4) Compute the odds of a full-house in 6-card poker using combinatorics.

note: your answer should match what you founding using enumeration.

hand = (xxx yy a) or (xxx yyy)

M1 = (xxx yy a)

Since the frequency of each value is different, do each one separately:

M1 = (13 values pick 1)(4 cards pick 3)(12 values pick 1)(4 cards pick 2)(11 values pick 1)

$$M_1 = \binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2}\binom{11}{1}\binom{4}{1}$$

$$M_1 = 164,736$$

M2 = (xxx yyy)

Since the frequency is the same for both variables, do them together

M2 = (13 choose 2 for xy)(4 choose 3 for x)(4 choose 3 for y)

$$M_2 = \binom{13}{2}\binom{4}{3}\binom{4}{3}$$

$$M_2 = 1248$$

The total is then

$$M = M_1 + M_2 = 165,984$$

The number of hands is

$$N = \binom{52}{6} = 20,358,520$$

This is the same answer we got in homework #1 using enumeration

```
Hands          Full-House   3-of-a-kind
20358520         165984          732160
```

5) Compute the odds being three of a kind using combinatorics

again, your answer should match what you founding using enumeration.

hand = xxx a b c

M = (13 values choose 1 for x)(4 cards choose 3)(12 values pick 3 for abc)(4 choose 1 for a)(4c1)(4c1)

$$M = \binom{13}{1}\binom{4}{3}\binom{12}{3}\binom{4}{1}\binom{4}{1}\binom{4}{1}$$

$$M = 732,160$$

This is the same answer we got using enumeration

```
Hands           Full-House  3-of-a-kind
20358520          165984         732160
```

6) Determine the odds of a full-house and three-of-a-kind using Matlab and a Monte-Carlo simulation and 100,000 hands of 6-card poker

Result with 100,000 hands:

```
     Full-House      3-of-a-kind
          876        3640

Elapsed time is 10.794453 seconds.
>>
```

	Enumeration	Calculated	Monte-Carlo
Full-House	815.30	815.30	876
3-of-a-kind 7	3596.33	3596.33	3,640

Expected Results with 100,000 Hands

Code:

```
% 6-Card Stud
tic
Pair32 = 0;
Pair3 = 0;

for i0 = 1:1e5

   X = rand(1,52);
   [a,Deck] = sort(X);
   Hand = Deck(1:6);
   Value = mod((Hand-1),13) + 1;
   Suit = ceil(Hand/13);

   N = zeros(1,13);
   for n=1:13
      N(n) = sum(Value == n);
      end

   [N,a] = sort(N, 'descend');

   if ((N(1) == 3)*(N(2) >= 2)) Pair32 = Pair32 + 1; end
   if ((N(1) == 3)*(N(2) < 2)) Pair3 = Pair3 + 1; end

end

clc
disp('    Full-House      3-of-a-kind');
disp([Pair32, Pair3]);
toc
```

Conditional Probability in 6-Card Poker

7) Compute the probability of getting a full-house if there is a single draw step

- If you are dealt 3-of-a-kind, draw 3 cards — hand = xxx abc discard abc, draw 3
- If you are dealt 2-pair, draw 2 cards — hand = xx yy ab discard ab, draw 2
- If you are dealt 1-pair, draw 4 cards — hand = xx abcd discard abcd, draw 4
- If you are dealt no pairs, draw 5 cards. — hand = abcdef discard abcde, draw 5

Note: This is actually a really hard problem since when you discard a two, there are fewer 2's left in the deck to pair up.

Assume that instead of discarding cards, they're shuffled back into the deck. The answer will be slightly high - but should be close.

Dealt 3 of a kind: p(A|B)

Ways to get to a full-house: Draw three cards

yyy + yya

M = (12c1 for y)(4c3 cards) + (12c1 for y)(4c2 cards)(11c1 for a)(4c1 card for a)

$$M = \binom{12}{1}\binom{4}{3} + \binom{12}{1}\binom{4}{2}\binom{11}{1}\binom{4}{1}$$

$$M = 3,216$$

$$N = \binom{49}{3} = 18,240$$ *assuming shuffle cards back into the deck*

$$p(A|B) = \tfrac{3216}{18240} = 0.176316$$

$$p(B) = 0.035963$$

$$p(A) = p(A|B)p(B) = 0.006341$$

634.1 in 100,000 hands

Dealt 2-Pair: (xx yy ab)

Ways to get to a full-house with drawing two cards

xy + xc + yc

M = (2c1 for x)(2c1 for y) + (2c1 for x)(44c1 for c) + (2c1 for y)(44c1 for c)

$$M = \binom{2}{1}\binom{2}{1} + \binom{2}{1}\binom{44}{1} + \binom{2}{1}\binom{44}{1}$$

$$M = 180$$

$$N = \binom{48}{2} = 1128$$ *assuming cards are shuffled back into the deck*

$$p(A|B) = \tfrac{M}{N} = \tfrac{180}{1128} = 0.159574$$

N(B) = xx yy ab + xx yy aa

(13c2 for xy)(4c2 for x)(4c2 for y)(11c2 for ab)(4c1 for a)(4c1 for b)

$$p(B) = \frac{\binom{13}{2}\binom{4}{2}\binom{4}{2}\binom{11}{2}\binom{4}{1}\binom{4}{1} + \binom{13}{3}\binom{4}{2}\binom{4}{2}\binom{4}{2}}{\binom{52}{6}}$$

$$p(B) = 0.12441$$

$$p(A) = p(A|B)p(B) = 0.019853$$

1985.3 in 100,000 hands

Dealt 1-pair (xx bcde)

Draw four cards

x yyy + x yy a + yyy a

(2c1 for x)(12c1 for y)(4c3) + (2c1 for x)(12c1 for y)(4c2)(44c1 for a) + (12c1 for y)(4c3)(44c1 for a)

$$M = \binom{2}{1}\binom{12}{1}\binom{4}{3} + \binom{2}{1}\binom{12}{1}\binom{4}{2}\binom{44}{1} + \binom{12}{1}\binom{4}{3}\binom{44}{1}$$

$$M = 8544$$

$$N = \binom{50}{4} = 230,300$$

$$p(A|B) = \tfrac{M}{N} = 0.037099$$

p(1 pair)

N(B) = (13 choose 1 for a)(4 choose 2)(12 choose 4 for bcde)(4c1)(4c1)(4c1)(4c1)

$$p(B) = \frac{\binom{13}{1}\binom{4}{2}\binom{12}{4}\binom{4}{1}\binom{4}{1}\binom{4}{1}\binom{4}{1}}{\binom{52}{6}}$$

$$p(B) = 0.485505$$

p(A)

$$p(A) = p(A|B)p(B)$$

$$p(A) = 0.018012$$

1801.2 in 100,000 hands

High Card: About the same as drawing a full house

$$p(A|B) \approx 0.008153$$

p(no pairs)

N=(13 choose 6)(4c1)(4c1)(4c1)(4c1)(4c1)(4c1)

$$p(B) = \frac{\binom{13}{6}\binom{4}{1}\binom{4}{1}\binom{4}{1}\binom{4}{1}\binom{4}{1}\binom{4}{1}}{\binom{52}{6}}$$

$$p(B) = 0.345284$$

$$p(A|B)p(B) = 0.002815$$

281.5 hands out of 100,000

Adding them all together....

Dealt Hand	calculated	Mont-Carlo (a)	Monte-Carlo (b)	Mont-Carlo (c)
full-house	815.3	811.1	809.1	805.3
3 of a kind	634.1	629.9	625.5	636.7
2 pair	1985.3	2068.3	2079.7	2096.8
pair	1801.2	1892.5	1863.4	1876.1
high card	281.5	306.4	311.1	312.8
Total	5517.4	5708.2	5688.8	5727.7

Calculated & Monte-Carlo: Number of hands out of 100,000 resulting in a full-house in 6-card draw poker
(Monte Carlo simulation actually used 1,000,000 hands)

The odds are fairly close - calculations are a little off due to assuming discards are shuffled back into the deck (wrong but simplifies calculations and is almost correct).

8) Check your answers using a Monte Carlo simulation with 100,000 hands of 6-card draw poker

see table in problem #7

note: In my Monte-Carlo program, I kept track of which path was taken to get to a full-house so I could check my calculations for each type of hand dealt.