Signals and Systems

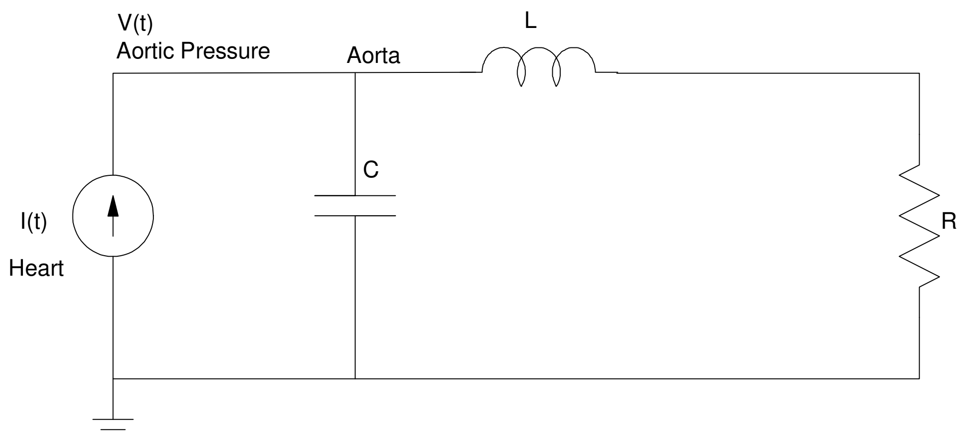
Fourier Transform: Solving differential equations when the input is periodic

Objectives

- Determine a least-squares curve fit for a function using a sinusoidal series (Fourier series)
- Determine the output of a filter when the input is periodic (but not a sine wave)

Problem:

One model for the cardiovascular system is a Winkessel model:



Assume you know I(t) (corresponding to the blood the heart pumps out). Find V(t).

Using s-notation, the impedance of R, L, and C become

$$R \to R$$

$$L \to j\omega L = Ls$$

$$C \to \frac{1}{j\omega C} = \frac{1}{Cs}$$

The resistance of the above circuit is then

$$Z = \left(\frac{1}{1/Cs} + \frac{1}{Ls+R}\right)^{-1}$$

$$Z = \left(Cs + \frac{1}{Ls+R}\right)^{-1}$$

$$Z = \left(\frac{Cs(Ls+R)+1}{Ls+R}\right)^{-1}$$

$$Z = \left(\frac{Ls+R}{CLs^2+CRs+1}\right)$$

Meaning

$$V = IR$$

$$V = \left(\frac{Ls+R}{CLs^2+CRs+1}\right) I$$

If you assume

- R = 1
- L = 0.1
- C = 1

you get

$$V = \left(\frac{s+10}{s^2+10s+10}\right) I$$

Case 1: Sinusoidal Input

Find V(t) assuming

$$I(t) = 3\sin(10t)$$

This is a phasor problem. Using phasor notation, voltages and currents are represented as

$$a + jb \rightarrow a\cos(\omega t) - b\sin(\omega t)$$

so

$$I(t) = 0 - j3$$

The frequency is 10 rad/sec

$$s = j10$$

so the output is then

$$V = \left(\frac{s+10}{s^2+10s+10}\right) I$$

$$V = \left(\frac{s+10}{s^2+10s+10}\right)_{s=j10} \cdot (-j3)$$

$$V = (0.0055 - j0.1050) \cdot (-j3)$$

$$V = \text{-0.3149 - 0.0166i}$$

meaning

$$\boxed{v(t) = -0.3149\cos(10t) + 0.0166\sin(10t)}$$

or if you prefer polar form

$$V = 0.3154\angle -177^0$$

meaning

$$v(t) = 0.3154\cos(10t - 177^0)$$

Either answer is valid.

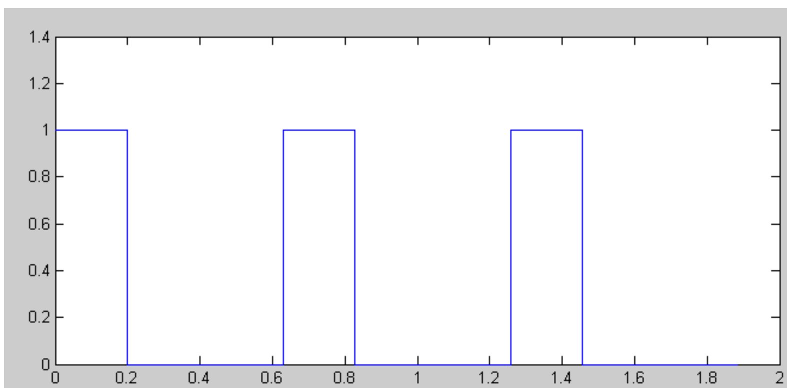
Case 2: Input is Periodic but Not a Sinusoid

Suppose instead I(t) is periodic every $\frac{2\pi}{10}$ seconds

$$I\left(t + \frac{2\pi}{10}\right) = I(t)$$

and

$$I(t) = \begin{cases} 1 & 0 < t < 0.2 \\ 0 & otherwise \end{cases}$$



I(t) vs. time. Note that I(t) is period every $\frac{2\pi}{10}$ seconds

Find V(t) assuming I(t) and V(t) are related by

$$V = \left(\frac{s+10}{s^2+10s+10}\right) I$$

Now find V(t)

Solution 1: Least Squares Curve Fitting:

To solve this problem, change it so that I(t) is a sinusoid, or more accurately, a bunch of sinusoids. Then you can use the previous solution for each sinusoidal input. Since I(t) is periodic in $\frac{2\pi}{10}$ seconds, express I(t) in terms of sine waves which are also periodic in $\frac{2\pi}{10}$ seconds (i.e. harmonics)

$$I(t) \approx a_0 + a_1 \cos(10t) + b_1 \sin(10t) + a_2 \cos(20t) + b_2 \sin(20t) + \ldots$$

In theory, you should go out to infinity. For the sake of space, let's just go out to 5 terms.

Using least squares, we can solve for the constants.

First, write this in matrix form

$$I(t) \approx \left(1 \;\; \cos(10t) \;\; \sin(10t) \;\; \cos(20t) \;\; \sin(20t) \right) \begin{bmatrix} a_0 \\ a_1 \\ b_1 \\ a_2 \\ b_2 \end{bmatrix}$$

or in matrix form

$$Y = BA$$

where B is your basis function and A is you unknow constants. B is not invertable, so multiply both sides by B^T

$$B^T Y = B^T B A$$

Solve for A

$$\boxed{A = (B^T B)^{-1} B^T Y}$$

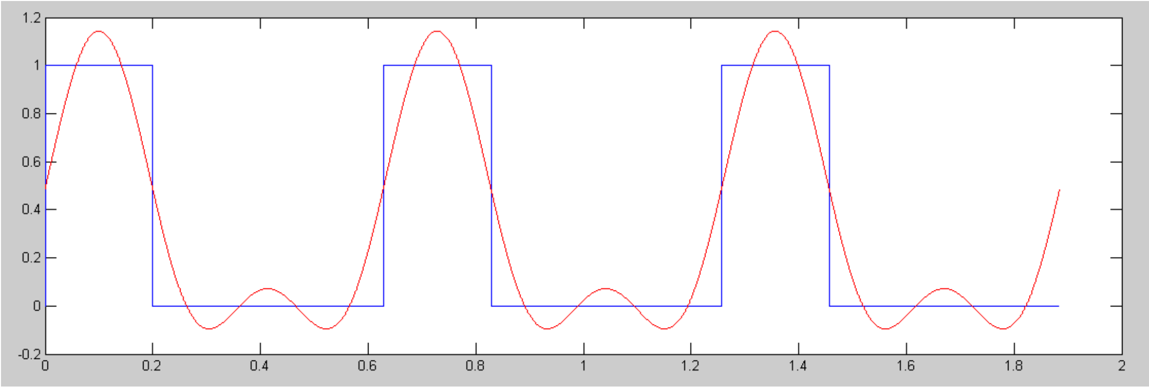
In Matlab:

```
>> t = [0:0.0001:1]' * 2*pi/10;
>> I = 1 .* (t < 0.2);

>> B = [t.^0, cos(10*t), sin(10*t), cos(20*t), sin(20*t)];
>> A = inv(B'*B)*B'*I

a0    0.3186
a1    0.2895
b1    0.4513
a2   -0.1208
b2    0.2634

>> plot(t,I,t,B*A);
```

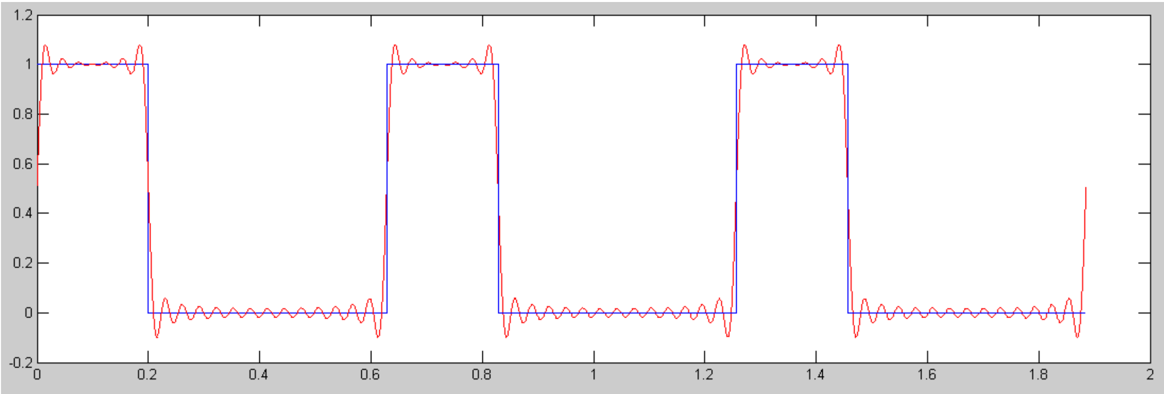


3-Cycles for I(t) (blue) and it's approximation using 5 terms (red): DC + 10 rad/sec + 20 rad/sec terms

What this means is

$$I(t) \approx 0.3186 + 0.2895\cos(10t) - 0.4513\sin(10t) - 0.1208\cos(20t) - 0.2634\sin(20t)$$

Note: The approximation gets better if you add more terms. If you go out to the 20th harmonic, this plot looks like the following:



3-Cycles for I(t) (blue) and it's approximation using 20 harmonics (40 terms + DC - red)

Adding more terms makes the analysis more accurate - but more tedious. It's kind of a judgement call: how many terms you need to include for the results to be accurate enough.

Often times, just a few harmonics are used:

- Signals tend to have most of their energy in the low-frequency terms (DC abnd1st harmonc typically)
- Most systems are low-pass filters: they tend to attenuate high-frequency signals.

This results in the higher harmonics starting out small and getting smaller after filtering. It looks like you need a lot of terms to do Fourier analysis - but in acuality you normally only need a few (1 to 3).

Solution 2: Fourier Transform

There's actually an easier way to get each term. Assume that

$$I(t) \approx a_0 + a_{10}\cos(10t) + b_{10}\sin(10t) + a_{20}\cos(20t) + b_{20}\sin(20t)$$

Note that

$$avg(\cos(at)) = 0$$

$$avg(\sin(at)) = 0$$

You can thus determine the DC term (a0) by

$$\boxed{a_0 = avg(I(t))}$$

Also note that all sine waves are orthogonal

$$avg(\sin(at)\cdot\cos(bt)) = 0$$

$$avg(\sin(at)\cdot\sin(bt)) = \begin{cases} \frac{1}{2} & a = b \\ 0 & otherwise \end{cases}$$

$$avg(\cos(at)\cdot\cos(bt)) = \begin{cases} \frac{1}{2} & a = b \\ 0 & otherwise \end{cases}$$

Thus

$$\boxed{\begin{aligned} a_{10} &= 2\cdot avg(\cos(10t)\cdot I(t)) \\ b_{10} &= 2\cdot avg(\sin(10t)\cdot I(t)) \end{aligned}}$$

$$\boxed{\begin{aligned} a_{20} &= 2\cdot avg(\cos(20t)\cdot I(t)) \\ b_{20} &= 2\cdot avg(\sin(20t)\cdot I(t)) \end{aligned}}$$

etc. In Matlab:

```
>> a0 = mean(I)

    0.3186

>> a10 = 2*mean(cos(10*t) .* I)

    0.2896

>> b10 = 2*mean(sin(10*t) .* I)

    0.4513

>> a20 = 2*mean(cos(20*t) .* I)
```

```
     -0.1207

>> b20 = 2*mean(sin(20*t) .* I)

      0.2634
```

Note that you get the exact same answer you got using least squares curve fitting. So, I(t) contains three frequencies (five terms):

- A DC term:

$$I_0(t) \approx 0.3186$$

- A term at 10 rad/sec

$$I_{10}(t) \approx 0.2895\cos(10t) - 0.4513\sin(10t)$$

- A term at 20 rad/sec

$$I_{20}(t) \approx 0.1208\cos(20t) - 0.2634\sin(20t)$$

In Matlab:

```
% Start with the DC term...
I0 = a0;

% add in the 10 rad/sec term...
I10 = a10*cos(10*t) + b10*sin(10*t);

% add in the 20 rad/sec term...
I20 = a20*cos(20*t) + b20*sin(20*t);

% plot I(t) alongs with it's approximation taken out to 20 rad/sec
plot(t,I,t,I0 + I10 + I20)
```

This gives same results as before.

	a0	a1	b1	a2	b2
Least Squares Solution	0.3186	0.2895	0.4513	-0.1208	0.2634
Fourier Transform Solution	0.3186	0.2896	0.4513	-0.1207	0.2634

Fourier Transforms is just curve fitting where you use sinusoids for the basis.

Finding V(t):

Using Fourier Transforms, you can convert a periodic signal into a sum of sinusoids. For our system

$$V = \left(\frac{s+10}{s^2+10s+10}\right) I$$

By using Fourier Transforms, I(t) can be expressed as

$$I(t) \approx 0.3186 + 0.2895\cos(10t) - 0.4513\sin(10t) - 0.1208\cos(20t) - 0.2634\sin(20t)$$

To find V(t), treat this as three separate problems: one at each frequency.

- $I_1(t) = 0.3186$
- $I_2(t) = 0.2895\cos(10t) - 0.4513\sin(10t)$
- $I_3(t) = -0.1208\cos(20t) - 0.2634\sin(20t)$

Phasor analuysis allows us to find V(t) for each separate input. Add up the results and you have the total outout, V(t).

To do this, set up a table: one column for each frequency. Analyze the circuit for each input separately

	I1(t)	**I2(t)**	**I3(t)**
Frequency: s	s = 0	s = j10	s = j20
i(t)	0.3186	0.2895 cos(10t) - 0.4513 sin(10t)	-0.1208 cos(20t) - 0.2634 sin(20t)
I(s): Phasor representation for i(t) note: a + jb means a cos(wt) - b sin(wt)	0.3186	0.2895 + j0.4513	-0.1201 + j0.2634
$G(s) = \left(\frac{s+10}{s^2+10s+10}\right)_{s=j\omega}$	1.000	0.0055 - j0.1050	0.0005 - j0.0510
V(s) = G(s) * I(s) phasor representation for v(t)	0.3186	0.0490 - j0.0279	0.0135 - j0.0060
v(t)	0.3186	0.0490 cos(10t) + 0.0279 sin(10t)	0.0135 cos(20t) - 0.0060 sin(20t)

If you add up all of the inputs (I1(t) + I2(t) + I3(t)), you get the total input.

If you add up all the outputs, you get the total output

$$v(t) = 0.3186 + 0.0490\cos(10t) + 0.0279\sin(10t) + 0.0135\cos(20t) - 0.0060\sin(20t)$$

Once you understand phasors, you can analyze any circuit or differential equation with sinusoidal inputs. Fourier Transfoms are a way to convert any periodic signal into a sum of sinusoidal inputs.

A step-by-step procedure to analyze this circuit at each frequency follows. It's really just filling in the above table...

Detailed Analysis at Each Frequency

Case 1:

$$I(t) = 0.3186$$

Using phasor analysis

$$I = 0.3186$$

$$s = 0$$

$$V = \left(\frac{s+10}{s^2+10s+10}\right)_{s=0} \cdot I$$

$$V = (1) \cdot (0.3186)$$

$$v(t) = 0.3186$$

Case 2:

$$I(t) = 0.2895\cos(10t) - 0.4513\sin(10t)$$

Using phasor analysis

$$I = 0.2895 + j0.4513$$

$$s = j10$$

$$V = \left(\frac{s+10}{s^2+10s+10}\right)_{s=j10} \cdot I$$

$$V = (0.0055 - j0.1050) \cdot (0.2895 + j0.4513)$$

$$V = -0.0458 - 0.0329i$$

$$v(t) = -0.0458\cos(10t) + 0.0329\sin(10t)$$

Case 3:

$$I(t) = 0.1208\cos(20t) - 0.2634\sin(20t)$$

Using phasor analysis

$$I = 0.1208 + j0.2634$$

$$s = j20$$

$$V = \left(\frac{s+10}{s^2+10s+10}\right)_{s=j20} \cdot I$$

$$V = (0.0005 - j0.0510) \cdot (0.1208 + j0.2634)$$

$$V = -0.0135 + j0.0060$$

$$v(t) = -0.0135\cos(20t) - 0.0060\sin(20t)$$

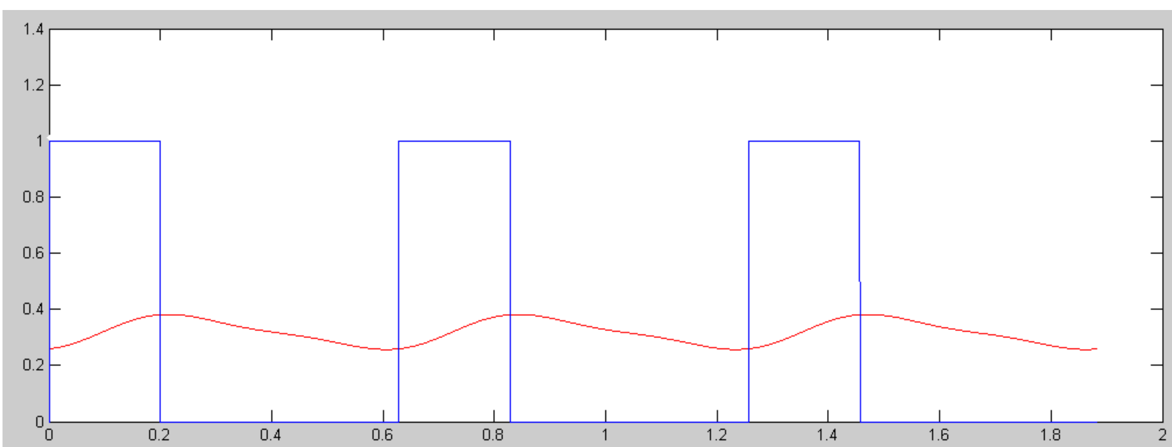
To get the total input, add up I(t) for part a), b), and c)

To get the total output, add up V(t) for part a), b), and c)

$$v(t) = v_0(t) + v_{10}(t) + v_{20}(t)$$

$$v(t) = 0.3186 - 0.0458\cos(10t) + 0.0329\sin(10t) - 0.0135\cos(20t) - 0.0060\sin(20t)$$

This looks like the following:



I(t) (blue) and V(t) (red) - shown for three cycles

Matlab Solution

In Matlab, this is actually a lot easier (hint: use this method for solving the homework)

First, input the function. Assume the period is $\frac{2\pi}{10}$

```
t = [0:0.0001:1]' * 2*pi/10;
x = 1 * (t < 0.2);
```

a) At s = 0 (DC),

```
s = 0;

a0 = mean(x);

I0 = a0

    0.3186              ( the DC component of I(t) )

G0 = ( s + 10 ) / ( s^2 + 10*s + 10 )

     1                  ( the gain at s = 0 )

V0 = G0 * I0

    0.3186              ( the DC component of V(t) )
```

b) At 10 rad/sec (1st harmonic)

```
s = j*10;

a10 = 2*mean(x .* cos(10*t));
b10 = 2*mean(x .* sin(10*t));

I10 = a10 - j*b10

   0.2896 - 0.4513i   ( the phasor representation for I(t) at 10 rad/sec )

G10 = ( s + 10 ) / ( s^2 + 10*s + 10 )

   0.0055 - 0.1050i   ( the gain at s = j10 )

V10 = G10 * I10

  -0.0458 - 0.0329i   ( the phasor representation for V(t) at 10 rad/sec )
```

c) At 20 rad/sec:

```
s = j*20;

a20 = 2 * mean( x .* cos(20*t) );
b20 = 2 * mean( x .* sin(20*t) );

I20 = a20 - j*b20

  -0.1207 - 0.2634i   ( the phasor representation for I(t) at 20 rad/sec )

G20 = ( s + 10 ) / ( s^2 + 10*s + 10 )

   0.0005 - 0.0510i   ( the gain at s = j20 )

V20 = G20 * I20

  -0.0135 + 0.0060i   ( the phasor representation for V(t) at 20 rad/sec )
```

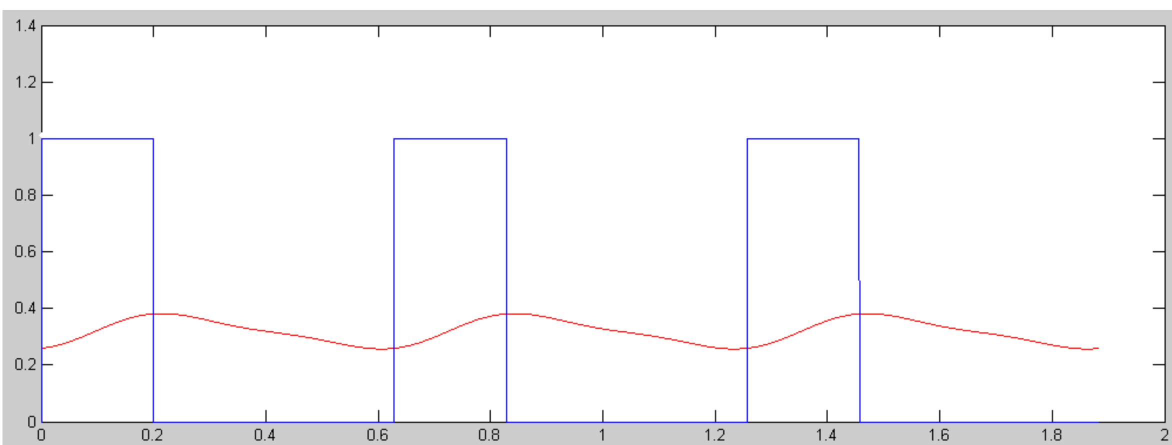
Finally, convert from phasor notoation back to time domain:

```
% Start with the DC term...
V = V0;

% add in the 10 rad/sec term...
V = V + real(V10)*cos(10*t) - imag(V10)*sin(10*t);

% add in the 20 rad/sec term...
V = V + real(V20)*cos(20*t) - imag(V20)*sin(20*t);

plot(t,I,t,V)
```



I(t) (blue) and V(t) (red) - shown for three cycles (same results as before)