ECE 111 - Homework #10

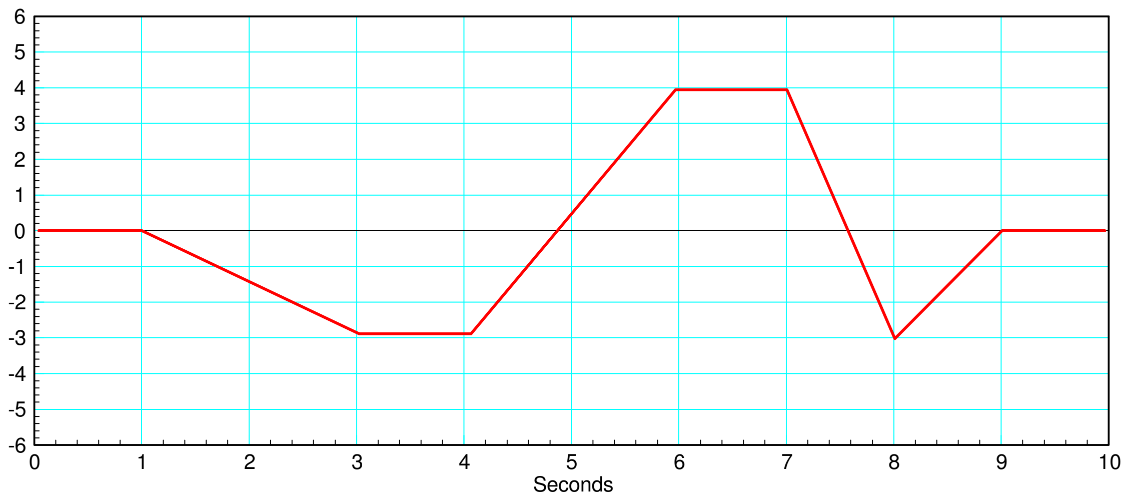
ECE 311 Circuits II - Capacitors & the Heat Equation

Due Monday, March 31st. Please submit via email or on BlackBoard

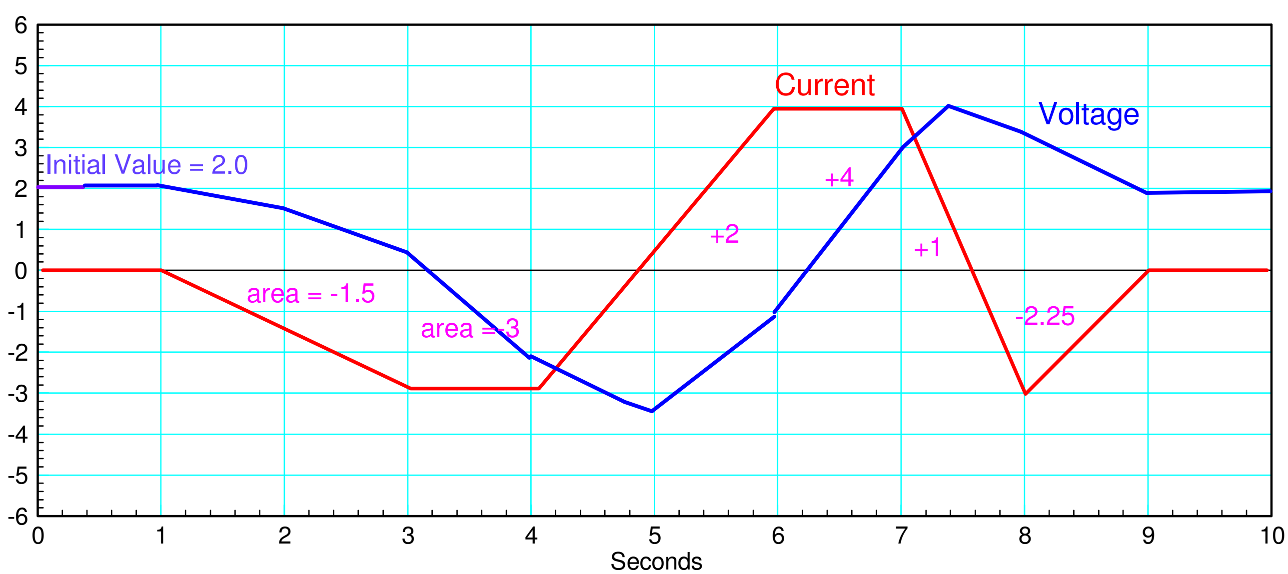
1) Assume the current flowing through a one Farad capacitor is shown below. Sketch the voltage. Assume V(0) = 0. The voltage is the integral of the current (capacitors are integrators)

$$V = \frac{1}{C}\int I \cdot dt$$

Assume the initial voltage on the capacitor is +2V.



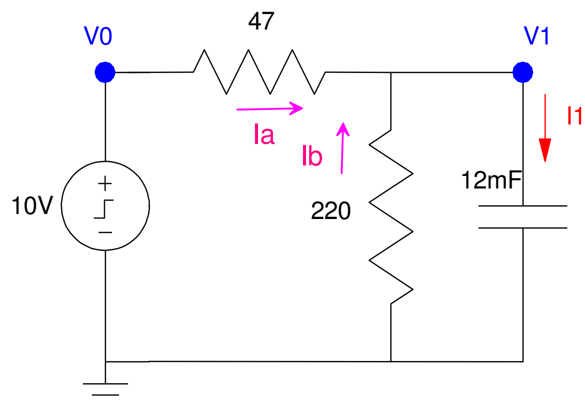
Capacitors integrate. The answer is the same as homework set #7



1-Stage RC filter:

2) Write the differential equation that describe this circuit. Note:

$$I_1 = C\frac{dV_1}{dt} = \sum(\text{current to node} V_1)$$



Current In = Current Out

$$I_1 = I_a + I_b$$

$$I_1 = 0.012\frac{dV_1}{dt} = \left(\frac{V_0 - V_1}{47}\right) + \left(\frac{0 - V_1}{220}\right)$$

Simplifying

$$\frac{dV_1}{dt} = 1.773V_0 - 2.190V_1$$

Comment:

- Writing the differential equations is fairly easy: just use conservation of current
  current in = current out
- Solving these differential equations by hand is something you'll do in Calculus III

For this course, we'll use Matlab to solve using numerical methods

- Once you get the differential equation, you pretty much have the program

3) Find and plot V1(t) for two seconds using Matlab.

Solve

$$\frac{dV_1}{dt} = 1.773V_0 - 2.190V_1$$

with the initial condition:

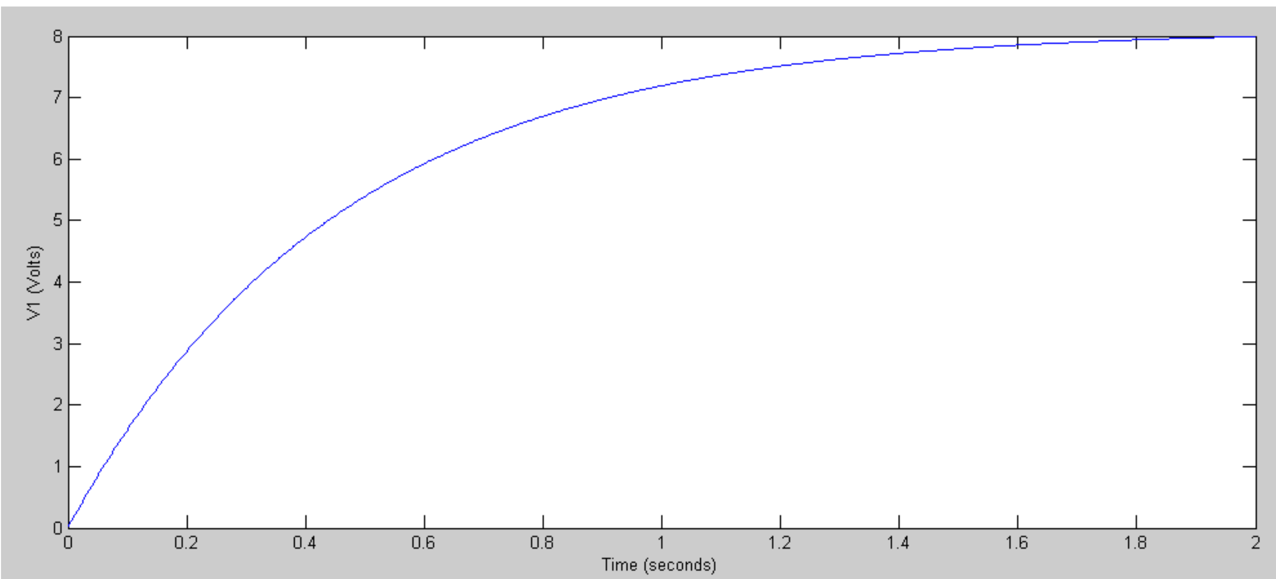
$$V_1(0) = 0$$

(Since V0 = 0 for time less than infinity, all voltages will be zero at t=0)

Write a Matlab program to simulate for 2 seconds.

- Set the integration step size to 2ms for 1000 points on the plot

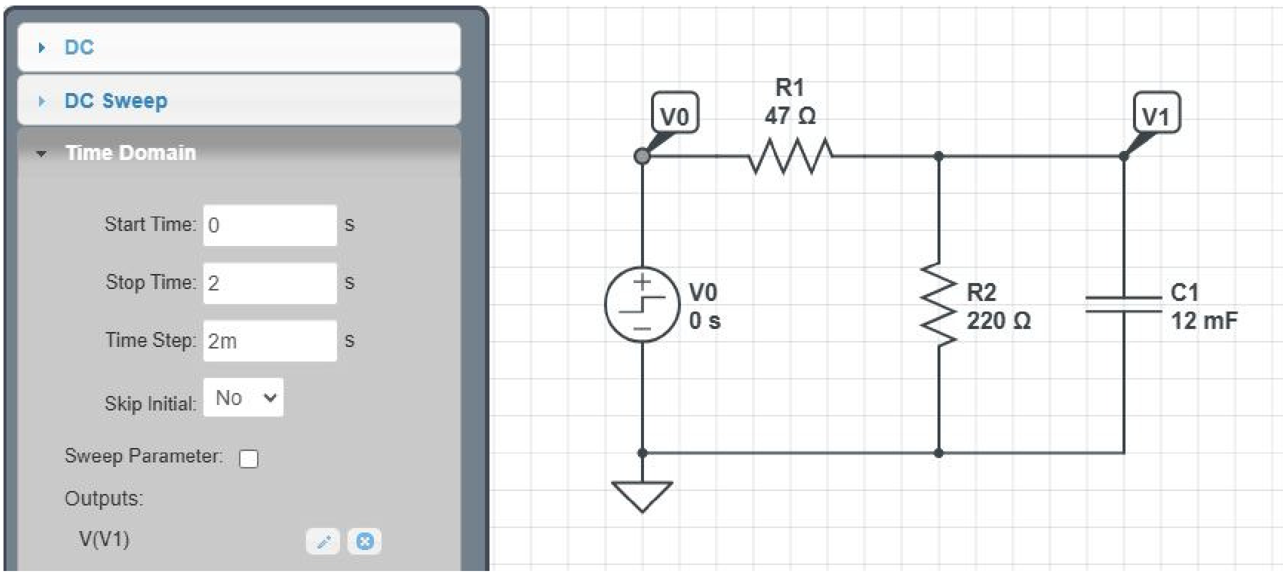
```
dt = 0.002;
t = 0;
V0 = 10;
V1 = 0;

x = [];
y = [];

while(t < 2)
    dV1 = 1.773*V0 - 2.190*V1;
    V1 = V1 + dV1 * dt;
    t = t + dt;
    x = [x; t];
    y = [y ; V1];
end

plot(x,y);
xlabel('Time (seconds)');
ylabel('V1 (Volts)')
```

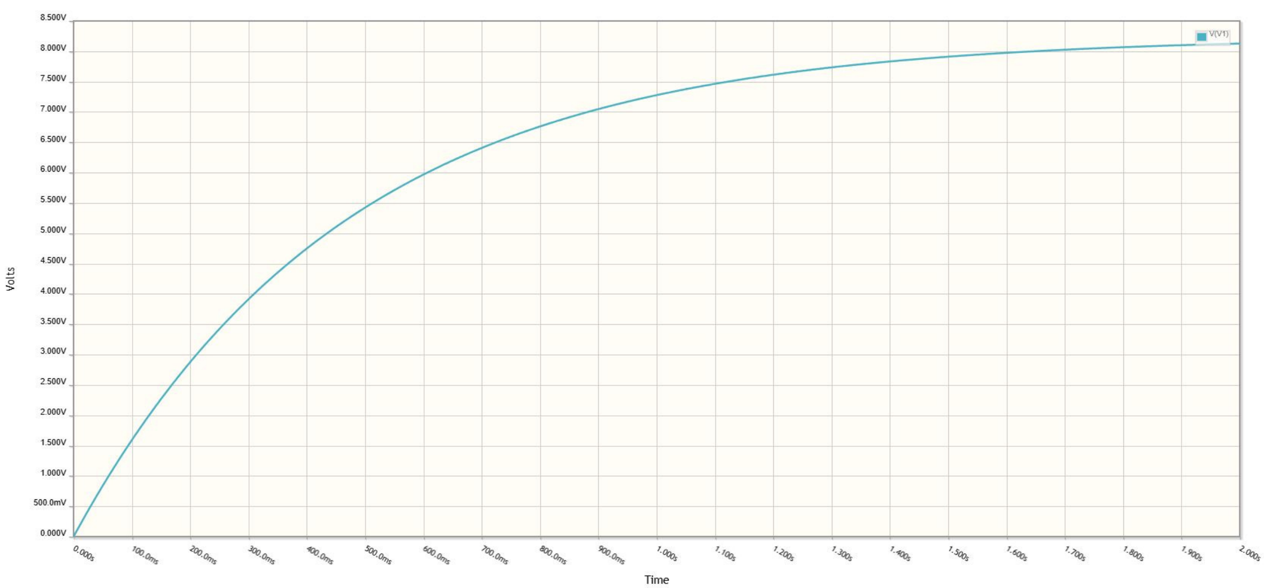


4) Find and plot V1(t) for two seconds using CircuitLab

- Set the simulating time to 2 seconds
- Set the simulation step size to 2ms (1000 points on the plot)

The graph is pretty much the same as we calculated in Matlab

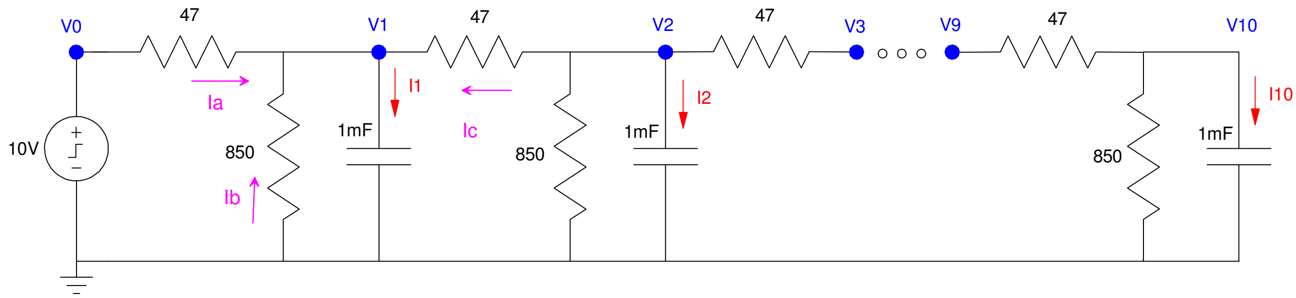




85010-Stage RC Filter

5) Write the dynamics for this system as a set of ten coupled differential equations:

$$I_1 = C\frac{dV_1}{dt} = \sum(\text{current to node} V_1)$$



Now, solve ten coupled 1st-order differential equations

- easy in Matlab
- hard by hand

First, write the differential equations. At node V1

$$I_1 = I_a + I_b + I_c$$

$$I_1 = 0.001\frac{dV_1}{dt} = \left(\frac{V_0 - V_1}{47}\right) + \left(\frac{0 - V_1}{850}\right) + \left(\frac{V_2 - V_1}{47}\right)$$

Simplifying

$$\frac{dV_1}{dt} = 21.277V_0 - 43.73V_1 + 21.277V_2$$

By symmetry, nodes 2..9 are all similar

$$\frac{dV_2}{dt} = 21.277V_1 - 43.73V_2 + 21.277V_3$$

$$\frac{dV_3}{dt} = 21.277V_2 - 43.73V_3 + 21.277V_4$$

$$\frac{dV_4}{dt} = 21.277V_3 - 43.73V_4 + 21.277V_5$$

$$\vdots$$

The odd-ball is node #10 since there is no current Ic

$$I_{10} = I_a + I_b$$

$$I_{10} = 0.001\frac{dV_{10}}{dt} = \left(\frac{V_9 - V_{10}}{47}\right) + \left(\frac{0 - V_{10}}{850}\right)$$

$$\frac{dV_{10}}{dt} = 21.277V_9 - 22.453V_{10}$$

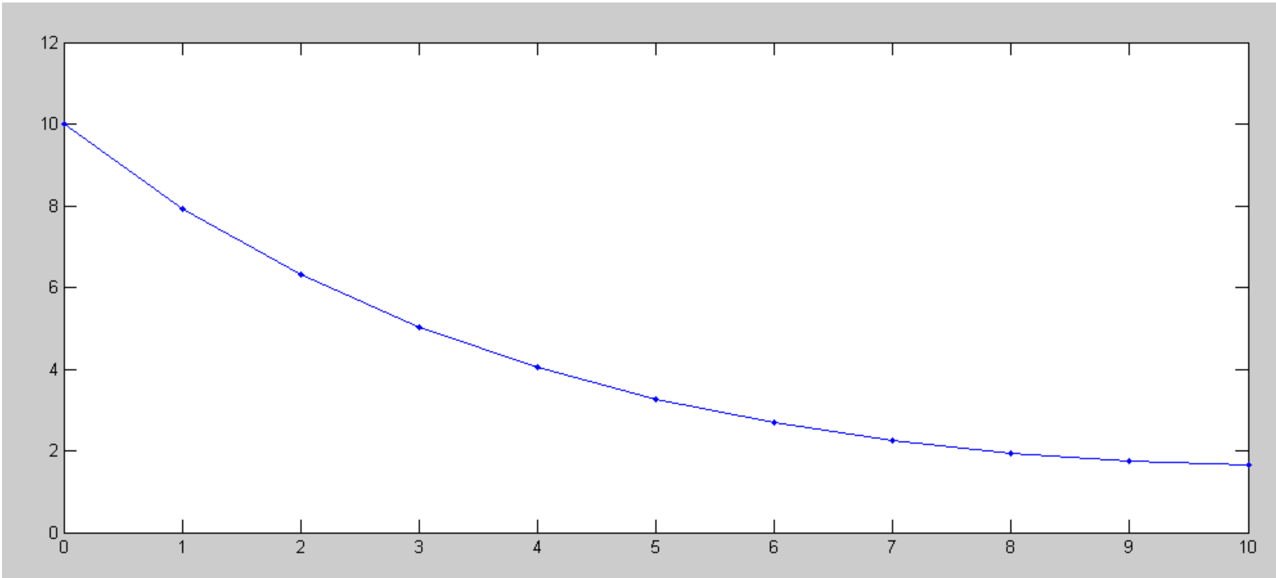
Forced Response for a 10-Node RC Filter (heat.m):

6) Using Matlab, solve these ten differential equations for $0 < t < 5$ s assuming

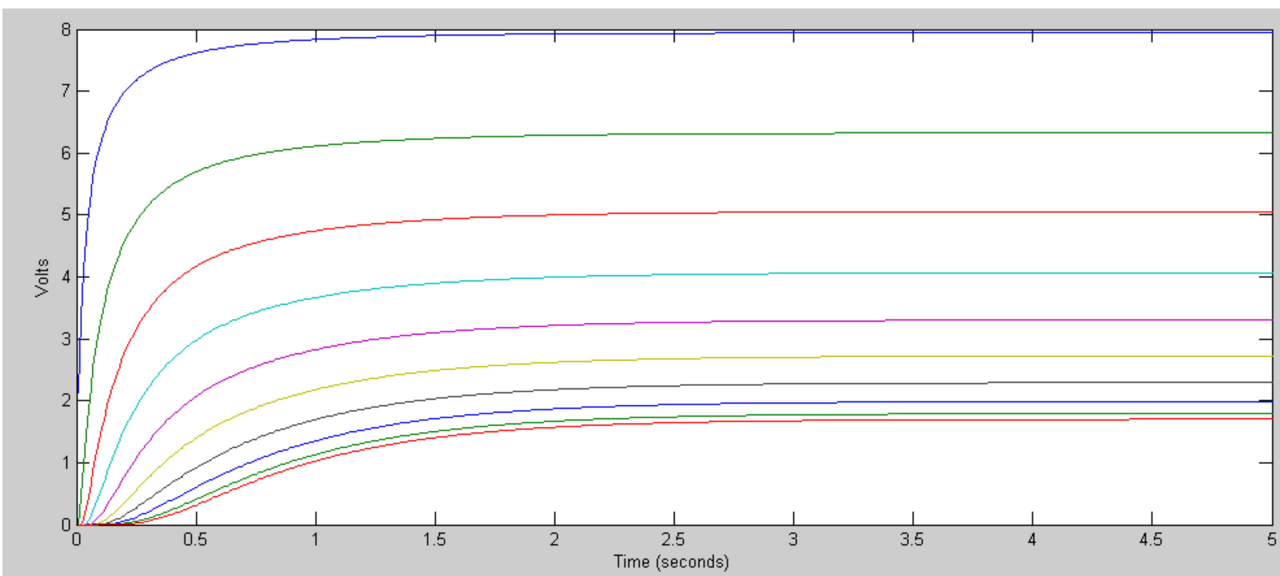
- The initial voltages are zero, and
- V0 = 10V.

Matlab Code:

```
V = zeros(10,1);
dV = 0*V;

dt = 0.01;
t = 0;
V0 = 10;

DATA = [];

while(t < 5)

   dV(1)  = 21.277*V0   - 43.73*V(1) + 21.277*V(2);
   dV(2)  = 21.277*V(1) - 43.73*V(2) + 21.277*V(3);
   dV(3)  = 21.277*V(2) - 43.73*V(3) + 21.277*V(4);
   dV(4)  = 21.277*V(3) - 43.73*V(4) + 21.277*V(5);
   dV(5)  = 21.277*V(4) - 43.73*V(5) + 21.277*V(6);
   dV(6)  = 21.277*V(5) - 43.73*V(6) + 21.277*V(7);
   dV(7)  = 21.277*V(6) - 43.73*V(7) + 21.277*V(8);
   dV(8)  = 21.277*V(7) - 43.73*V(8) + 21.277*V(9);
   dV(9)  = 21.277*V(8) - 43.73*V(9) + 21.277*V(10);
   dV(10) = 21.277*V(9) - 22.453*V(10);

   V = V + dV * dt;
   t = t + dt;

   plot([0:10], [V0;V], 'b.-');
   ylim([0,12]);
   xlim([0,10]);
   pause(0.01);
   DATA = [DATA ; V'];
end

pause(2);
t = [1:length(DATA)]' * dt;
plot(t,DATA)
```

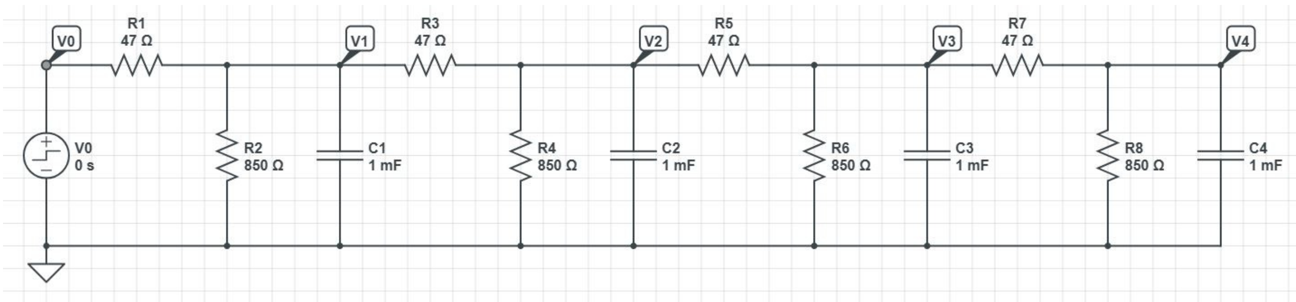


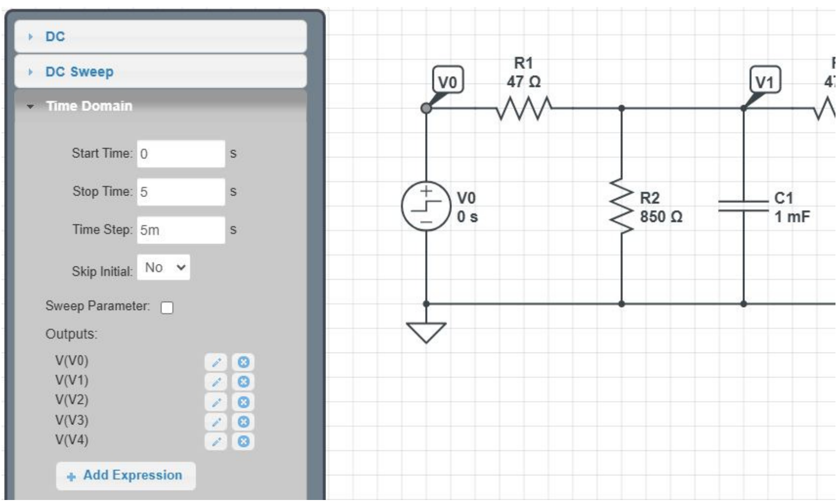
Node voltages at t = 5 seconds

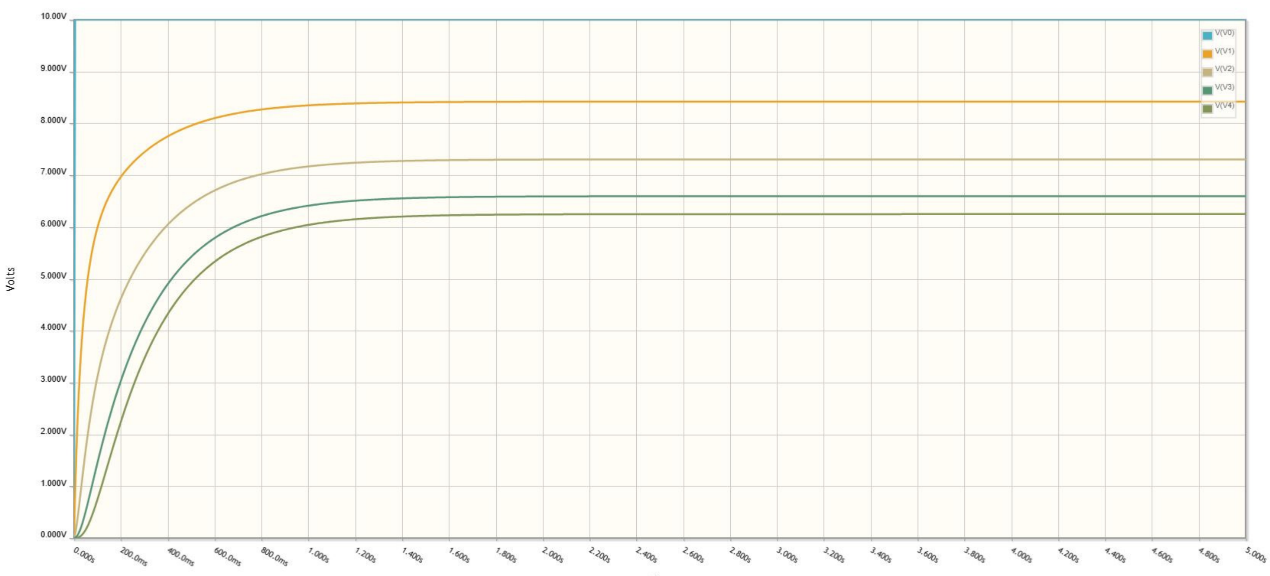


Voltages vs time

7) Using CircuitLab, find the response of this circuit to a 10V step input. *note: It's OK if you only build this circuit to 3 nodes...*







Natural Response: Eigenvectors and Eigenvalues

8) Assume V0 = 0V. Determine the initial conditions of V1..V10 so that

- The maximum voltage is 10V and
- 5a) The voltages go to zero as slow as possible
- 5b) The voltages go to zero as fast as possible.

Simulate the response for these initial conditions in Matlab.

This is an eigenvector problem. Experss the dynamics in matrix form

$$\frac{dV}{dt} = AV$$

From before

```
dV(1)  = 21.277*V0    - 43.73*V(1) + 21.277*V(2);
dV(2)  = 21.277*V(1)  - 43.73*V(2) + 21.277*V(3);
dV(3)  = 21.277*V(2)  - 43.73*V(3) + 21.277*V(4);
dV(4)  = 21.277*V(3)  - 43.73*V(4) + 21.277*V(5);
dV(5)  = 21.277*V(4)  - 43.73*V(5) + 21.277*V(6);
dV(6)  = 21.277*V(5)  - 43.73*V(6) + 21.277*V(7);
dV(7)  = 21.277*V(6)  - 43.73*V(7) + 21.277*V(8);
dV(8)  = 21.277*V(7)  - 43.73*V(8) + 21.277*V(9);
dV(9)  = 21.277*V(8)  - 43.73*V(9) + 21.277*V(10);
dV(10) = 21.277*V(9)  - 22.453*V(10);
```

In matrix form

$$dV = \begin{matrix} -43.73 & 21.28 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 21.28 & -43.73 & 21.28 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 21.28 & -43.73 & 21.28 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 21.28 & -43.73 & 21.28 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 21.28 & -43.73 & 21.28 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 21.28 & -43.73 & 21.28 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 21.28 & -43.73 & 21.28 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 21.28 & -43.73 & 21.28 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 21.28 & -43.73 & 21.28 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 21.28 & -22.453 \end{matrix} \quad V$$

This is a 10x10 matrix. It has

- Ten eigenvalues, and
- Ten eigenvectors

(something you'll cover when you take Math 129). With Matlab, you can calculate what these are:

In Matlab, input the 10x10 matrix. Since all terms repeat along the diagonal,

- Start with a 10x10 matrix of zeros (most terms are zero),
- Change the terms along the diagonal

A(10,10) is the odd-ball and is input by itself at the end

```
>> A = zeros(10,10);
>> for i = 1:9
    A(i,i) = -43.73;
    A(i+1,i) = 21.28;
    A(i,i+1) = 21.28;
    end
>> A(10,10) = -22.453
```

```
  -43.7300   21.2800         0         0         0         0         0         0         0         0
   21.2800  -43.7300   21.2800         0         0         0         0         0         0         0
         0   21.2800  -43.7300   21.2800         0         0         0         0         0         0
         0         0   21.2800  -43.7300   21.2800         0         0         0         0         0
         0         0         0   21.2800  -43.7300   21.2800         0         0         0         0
         0         0         0         0   21.2800  -43.7300   21.2800         0         0         0
         0         0         0         0         0   21.2800  -43.7300   21.2800         0         0
         0         0         0         0         0         0   21.2800  -43.7300   21.2800         0
         0         0         0         0         0         0         0   21.2800  -43.7300   21.2800
         0         0         0         0         0         0         0         0   21.2800  -22.4530
```

Now that A is in Matlab, find the eigenvalues and eivenvectors

```
>> [M,V] = eig(A)

Eigenvectors

    fast                                                                                   slow
   -0.1286   -0.2459    0.3412    0.4063    0.4352    0.4255    0.3780    0.2969   -0.1894   0.0651
    0.2459    0.4063   -0.4255   -0.2969   -0.0651    0.1894    0.3780    0.4352   -0.3412   0.1287
   -0.3412   -0.4255    0.1894   -0.1894   -0.4255   -0.3412   -0.0000    0.3412   -0.4255   0.1894
    0.4063    0.2969    0.1894    0.4352    0.1287   -0.3412   -0.3780    0.0650   -0.4255   0.2459
   -0.4352   -0.0651   -0.4255   -0.1287    0.4063    0.1894   -0.3779   -0.2459   -0.3412   0.2969
    0.4255   -0.1894    0.3412   -0.3412   -0.1894    0.4255    0.0000   -0.4255   -0.1893   0.3413
   -0.3780    0.3780   -0.0000    0.3780   -0.3780   -0.0000    0.3780   -0.3779    0.0001   0.3780
    0.2969   -0.4352   -0.3412    0.0650    0.2459   -0.4255    0.3779   -0.1286    0.1894   0.4063
   -0.1894    0.3412    0.4255   -0.4255    0.3412   -0.1893   -0.0000    0.1894    0.3413   0.4254
    0.0651   -0.1287   -0.1894    0.2459   -0.2969    0.3412   -0.3780    0.4063    0.4255   0.4351

Eigenvalues

  -84.3992  -78.8948  -70.2658  -59.2791  -46.9108  -34.2599  -22.4504  -12.5318   -5.3853  -1.6459
```

Eigenvalues tell you *how* the system behaves.

- This is a 10th-order system
- It has 10 energy states
- The fast mode decays as exp(-84.399t)
- The slow mode decays as exp(-1.645t)

Eigenvectors tell you *what* behaves that way

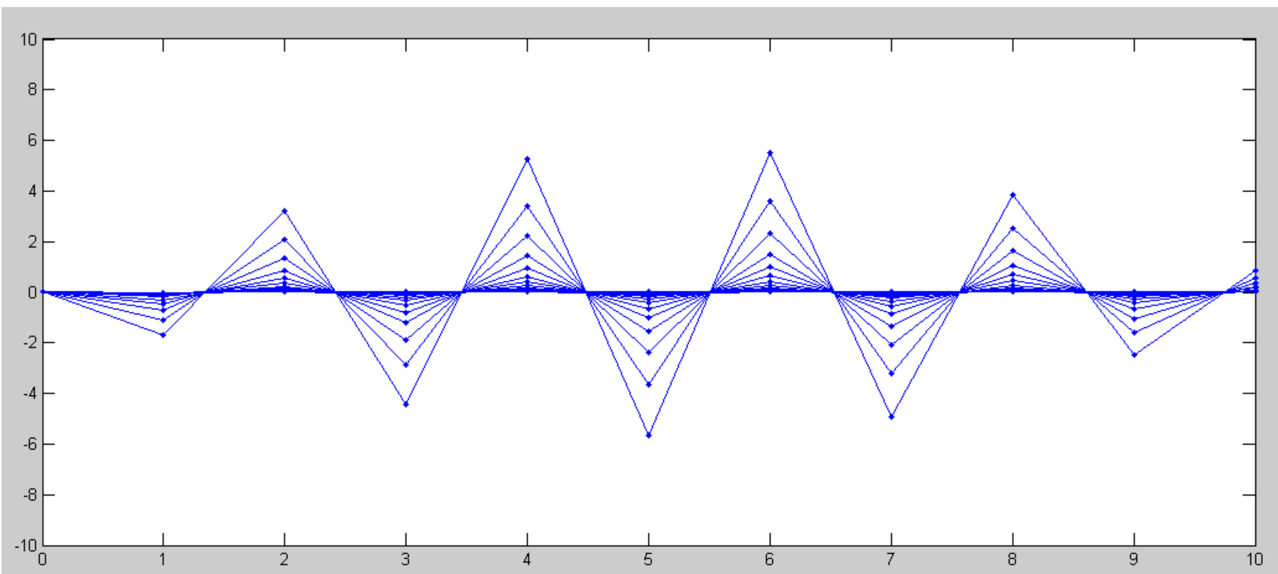
- If you make the initial conditions the fast eigenvector, the states decay as exp(-84.399t)
- If you make the initial conditions the slow eigenvector, the states decay as exp(-1.64t)

Try this in Matlab

Fast Mode:

```
V = 20*M(:,1);
dV = 0*V;

dt = 0.001;
t = 0;
V0 = 0;

DATA = [];

while(t < 0.5)

   dV(1)  = 21.277*V0   - 43.73*V(1) + 21.277*V(2);
   dV(2)  = 21.277*V(1) - 43.73*V(2) + 21.277*V(3);
```

This is the fast mode - it decays as exp(-84.39t)



The shape as it decays (show up better in the video)

- The shape remains the same (the shape is the eigenvector)

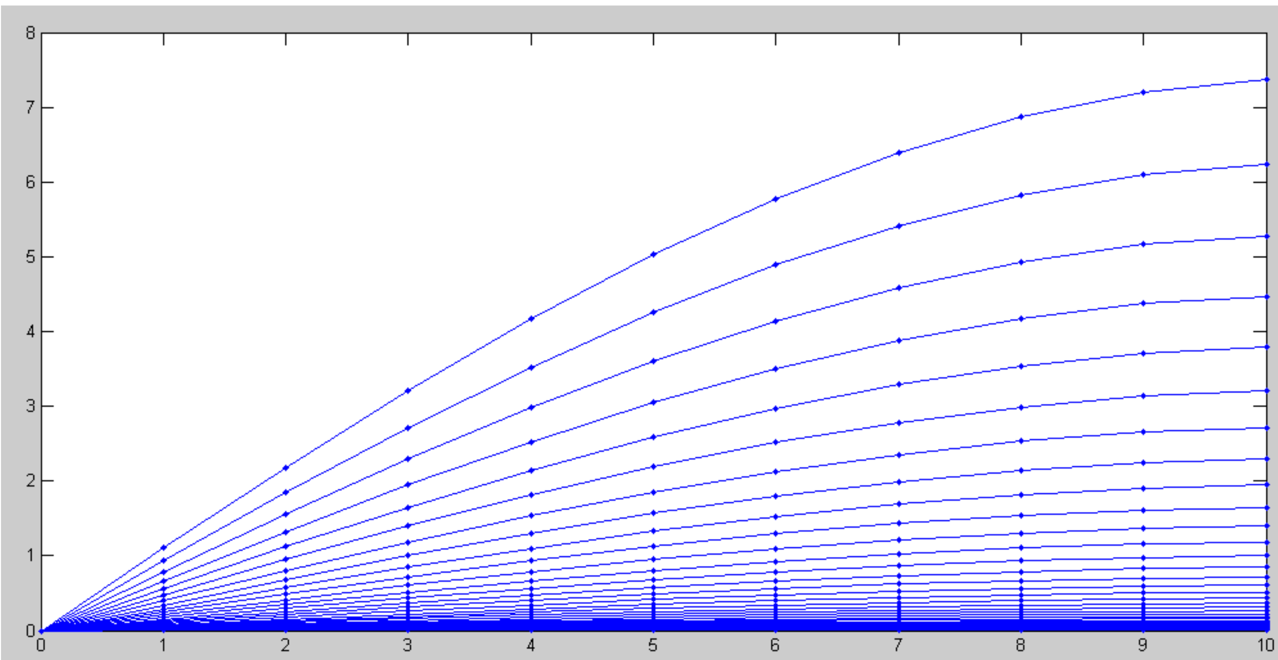
The amplitude decays as exp(-83.39t)

- The amplitude is the eigenvalue

Slow Mode

```
V = 20*M(:,10);
dV = 0*V;

dt = 0.01;
t = 0;
V0 = 0;

DATA = [];

while(t < 0.5)

   dV(1)  = 21.277*V0   - 43.73*V(1) + 21.277*V(2);
   dV(2)  = 21.277*V(1) - 43.73*V(2) + 21.277*V(3);
   :
```

This decays as exp(-1.64t)



Note the shape remains the same as it runs

- The shape is the eigenvector

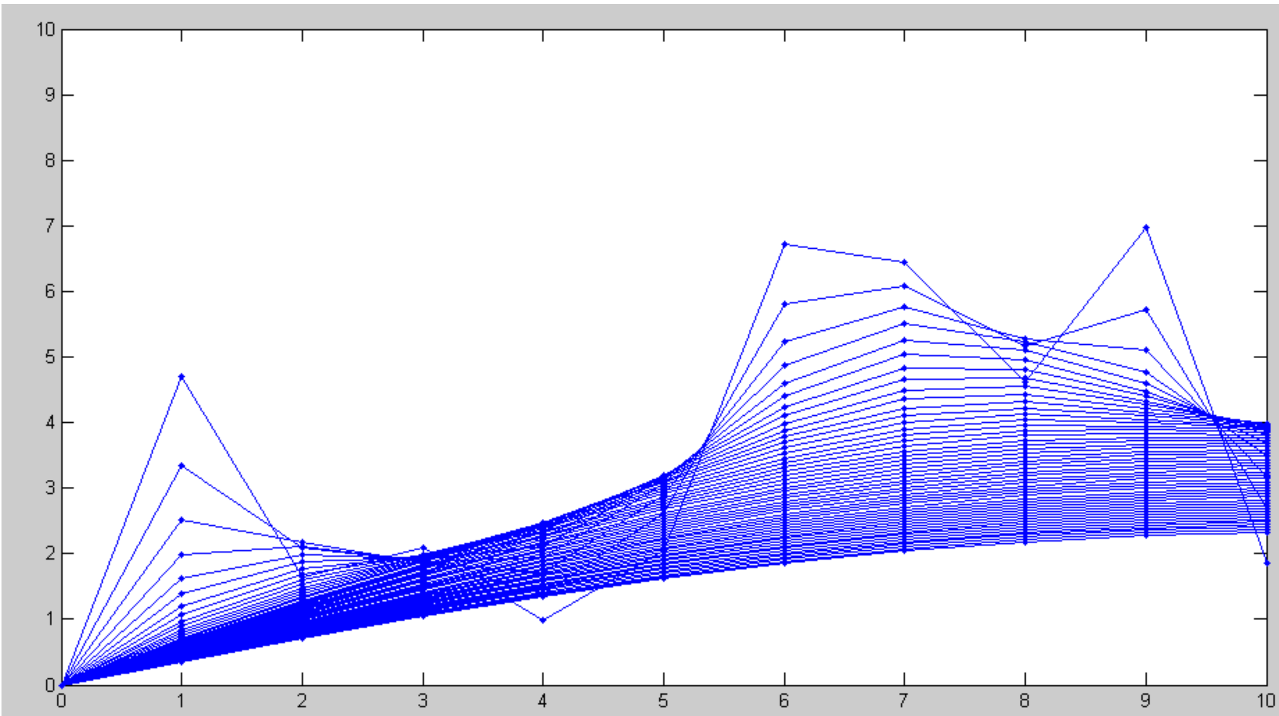
The amplitude decays as exp(-1.64t)

- The amplitude is the eivenvalue

9) Assume Vin = 0V. Pick random voltages for V1 .. V10 in the range of (0V, 10V):

```
V = 10 * rand(10,1)
```

Plot the votlages at t = 2. Which eigenvector does it look like?



Voltages plotted every 10ms

Random intial condistions excite all ten eigenvectors

- The fast modes decay quickly
- Leaving the slow (dominant) mode

After 2 seconds, all that's left is the slow eigenvector

Comment: Very few people understand eigenvalues. Even fewer understand eigenvectors.

It helps to have something concrete to refer to when trying to understand a difficult concept.

When you take Linear Algebra, you'll cover eigenvalues and eigenvectors - and most of the class will be lost. When you get to Linear Algebra, think about this problem and the heat equation. Just remember:

- *Eigenvalues tell you how a system behaves (how each mode behaves)*
- *Eigenvectors tell you what behaves that way (the shape of the modes)*

If you make the intial conditions an eigenvector, only that one mode is excited

If you make the initial condistions something random, all of the modes get excited

- *The fast modes decay quickly, the slow modes decay slowly*

When you get to Linear Algebra and need to find eigenvalues and eigenvectors, think Matlab.