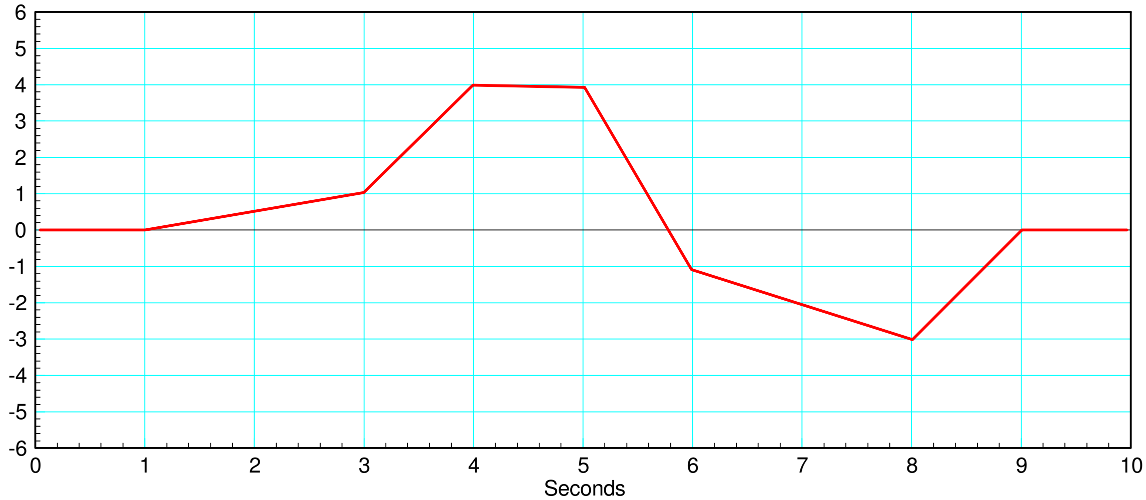
ECE 111 - Homework #10

ECE 311 Circuits II - Heat Equation

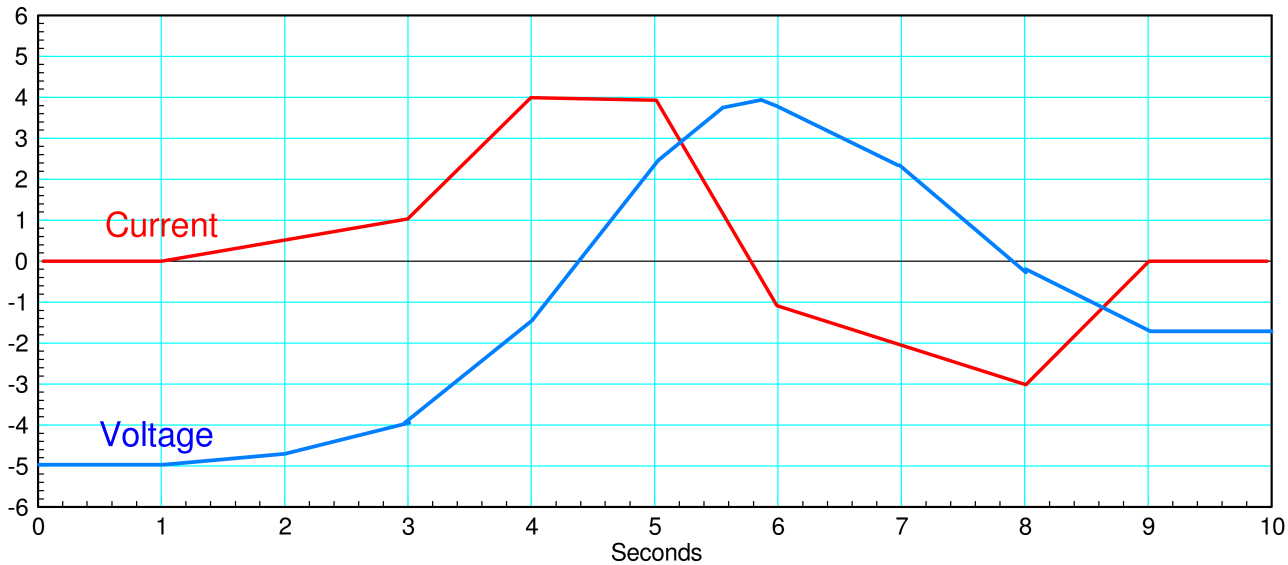
1) Assume the current flowing through a one Farad capacitor is shown below. Sketch the voltage. Assume V(0) = 0. The voltage is the integral of the current (capacitors are integrators)

$$V = \frac{1}{C}\int I \cdot dt$$

Assume the initial voltage on the capacitor is -5V.



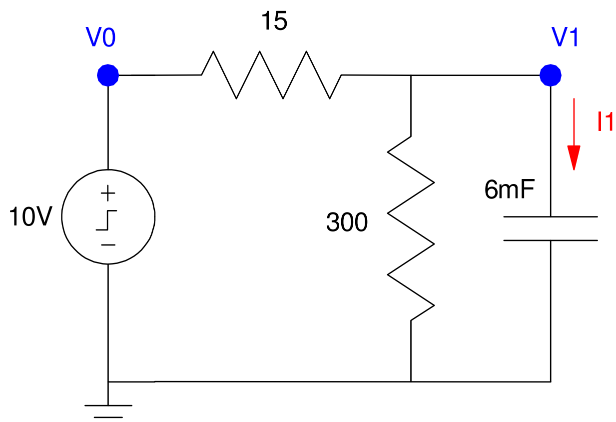
Capacitor are integrators: the voltage is the integral of the current (the area under the curve)



1-Stage RC filter:

2) Write the differential equation that describe this circuit. Note:

$$I_1 = C\frac{dV_1}{dt} = \sum(\text{current to node}V_1)$$



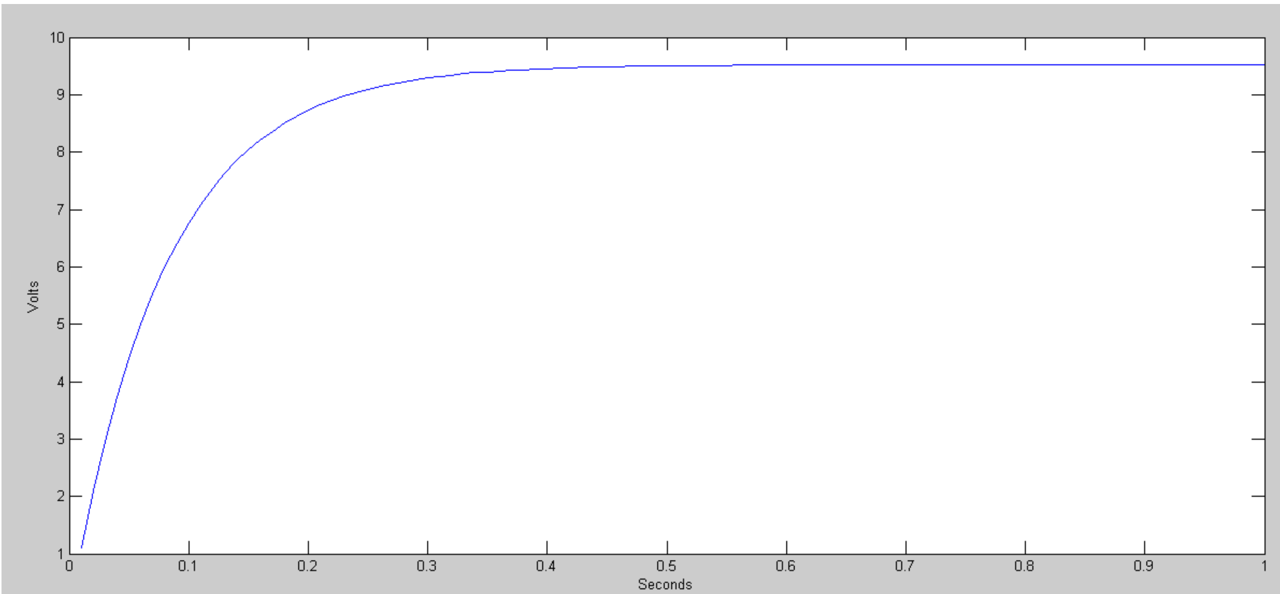
$$I_1 = 0.006 \cdot \frac{dV_1}{dt} = \left(\frac{V_0 - V_1}{15}\right) - \left(\frac{V_1}{300}\right)$$

$$\frac{dV_1}{dt} = 11.11V_0 - 11.6676V_1$$

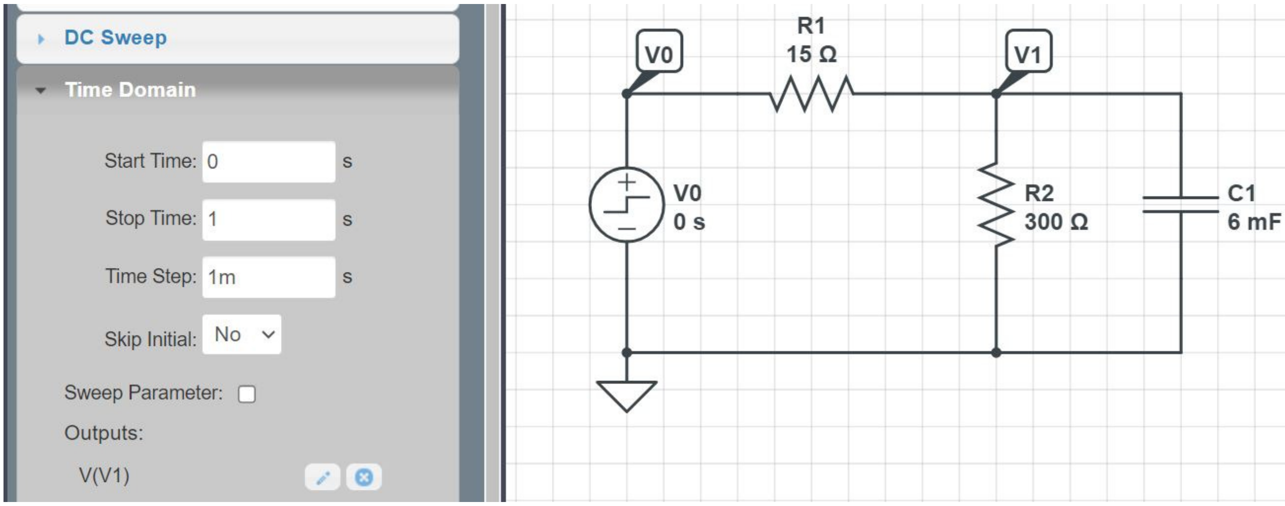
3) Find and plot V1(t) for one second using Matlab.

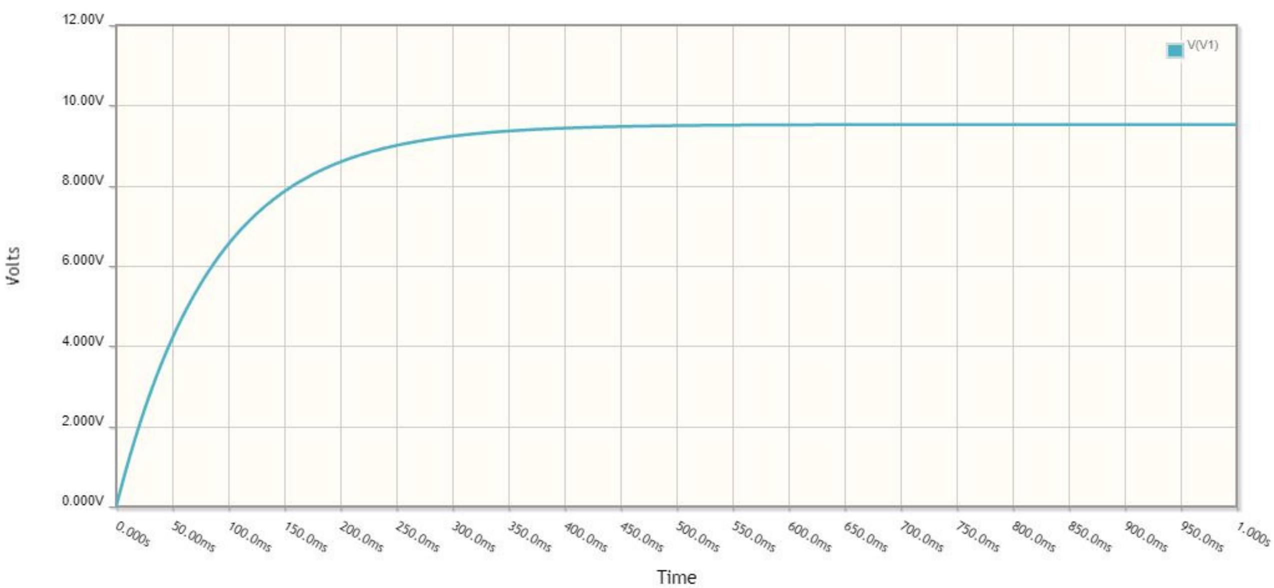
Matlab Code

```
V1 = 0;
t = 0;
dt = 0.01;
y = [];
V0 = 10;

while(t < 1)
    dV1 = 11.11*V0 - 11.667*V1;

    V1 = V1 + dV1*dt;
    t = t + dt;

    y = [y ; V1];
end

t = [1:length(y)]' * dt;
plot(t,y);
xlabel('Seconds');
ylabel('Volts');
```



4) Find and plot V1(t) for one seconds using CircuitLab



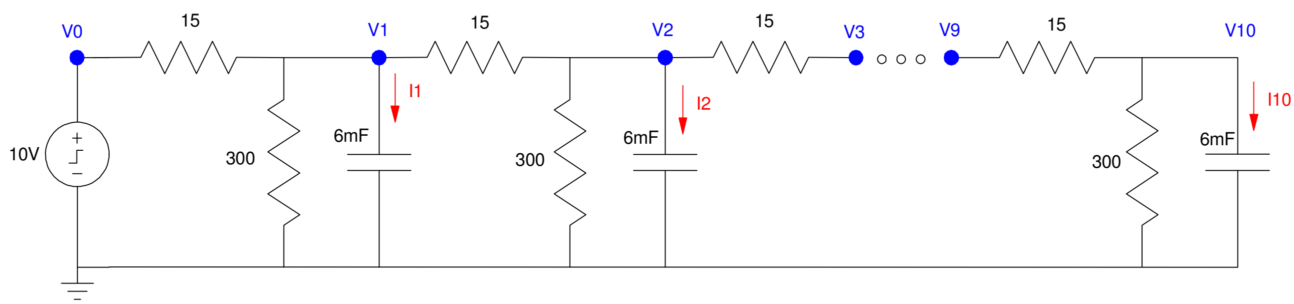


Note: This is the same result we got with Matlab

10-Stage RC Filter

5) Write the dynamics for this system as a set of ten coupled differential equations:

$$I_1 = C\frac{dV_1}{dt} = \sum(\text{current to node} V_1)$$



At node V1:

$$C \cdot \frac{dV_1}{dt} = \left(\frac{V_0 - V_1}{15}\right) + \left(\frac{V_2 - V_1}{15}\right) - \left(\frac{V_1}{300}\right)$$

Simplifying

$$\frac{dV_1}{dt} = \left(\frac{1}{15C}\right)V_0 - \left(\frac{1}{15C} + \frac{1}{15C} + \frac{1}{300C}\right)V_1 + \left(\frac{1}{15C}\right)V_2$$

$$\frac{dV_1}{dt} = 11.111V_0 - 22.778V_1 + 11.111V_2$$

The same pattern holds for nodes 2..9

$$\frac{dV_2}{dt} = 11.111V_1 - 22.778V_2 + 11.111V_3$$

$$\vdots$$

$$\frac{dV_9}{dt} = 11.111V_8 - 22.778V_9 + 11.111V_{10}$$

The last node is a little different since there is only a single 15 ohm resistor connected

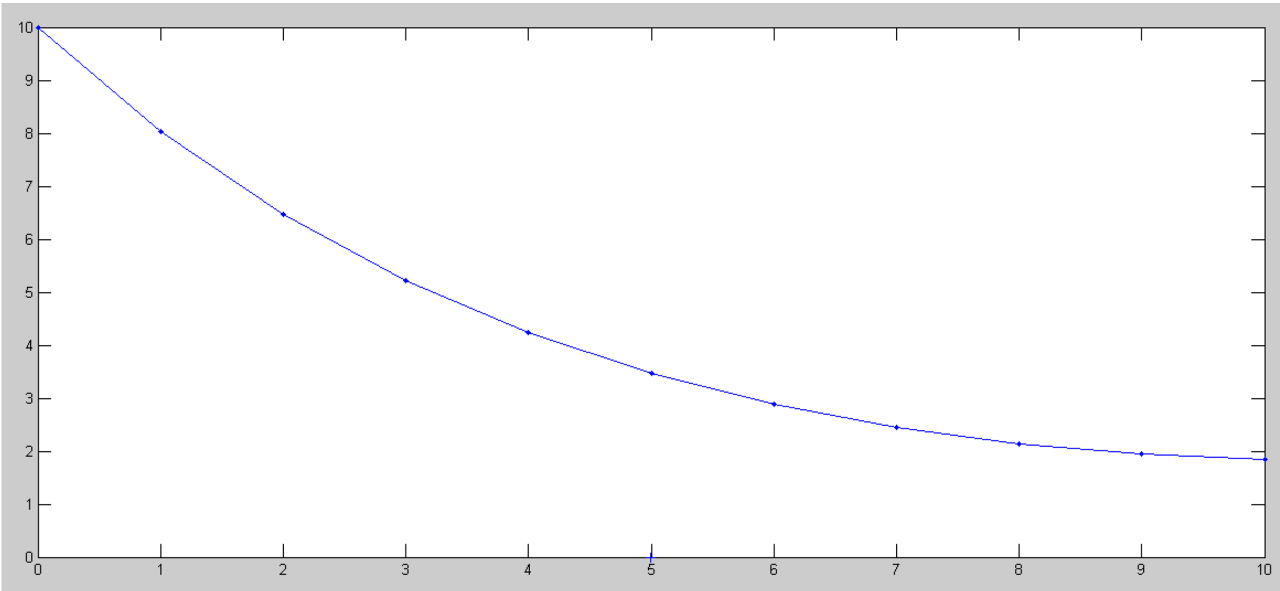
$$\frac{dV_{10}}{dt} = \left(\frac{1}{15C}\right)V_9 - \left(\frac{1}{15C} + \frac{1}{300C}\right)V_{10}$$

$$\frac{dV_{10}}{dt} = 11.111V_9 - 11.667V_{10}$$

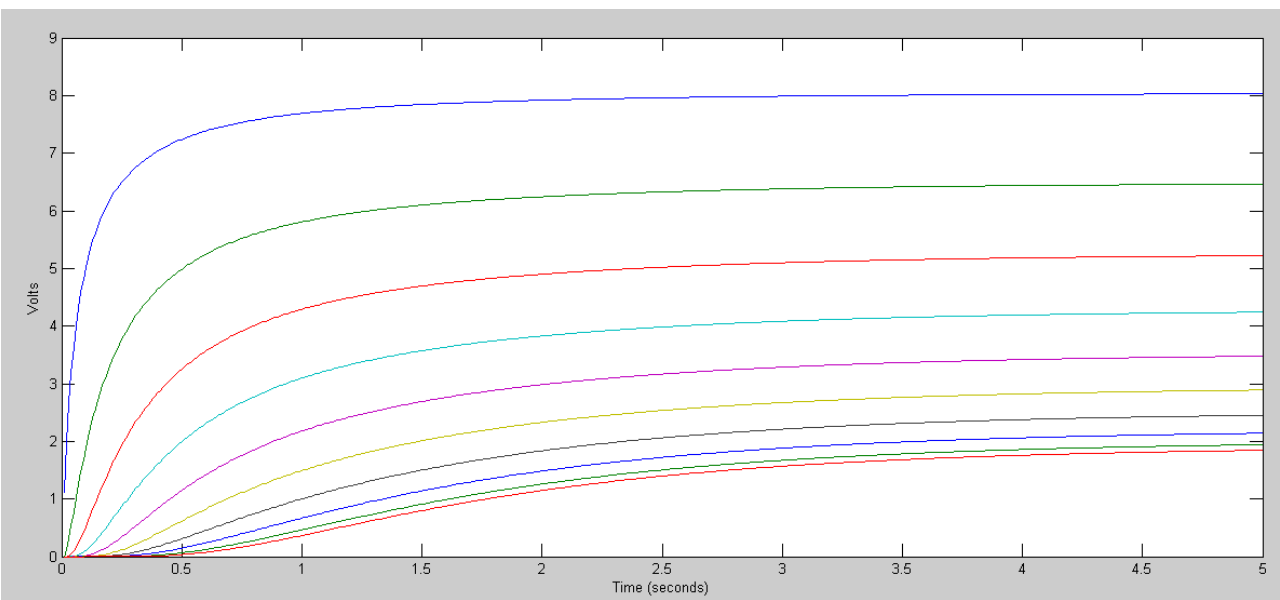
Forced Response for a 10-Node RC Filter (heat.m):

6) Using Matlab, solve these ten differential equations for $0 < t < 5$ s assuming

- The initial voltages are zero, and
- $V0 = 10V$.



Voltages at t -= 5 seconds



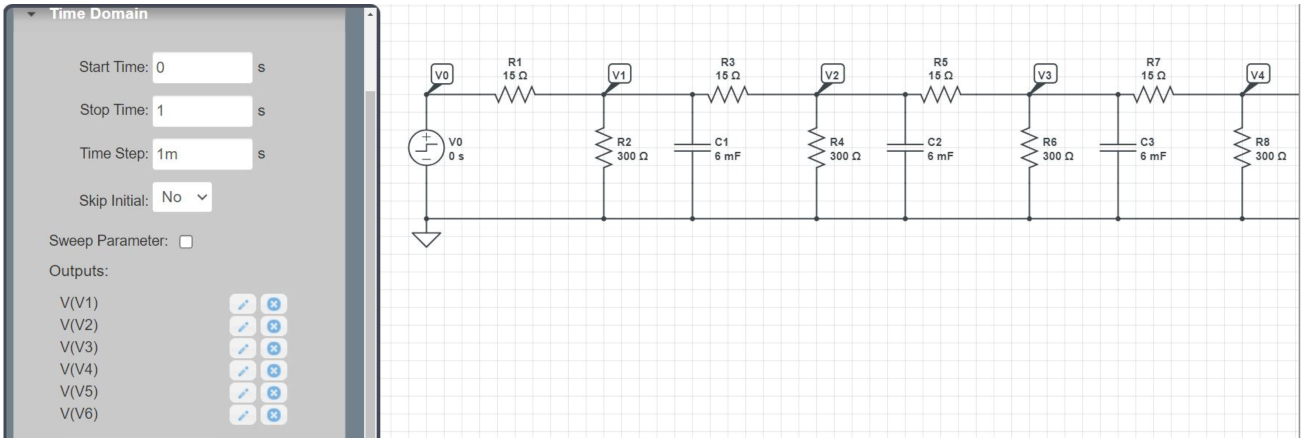
Voltages vs. Time

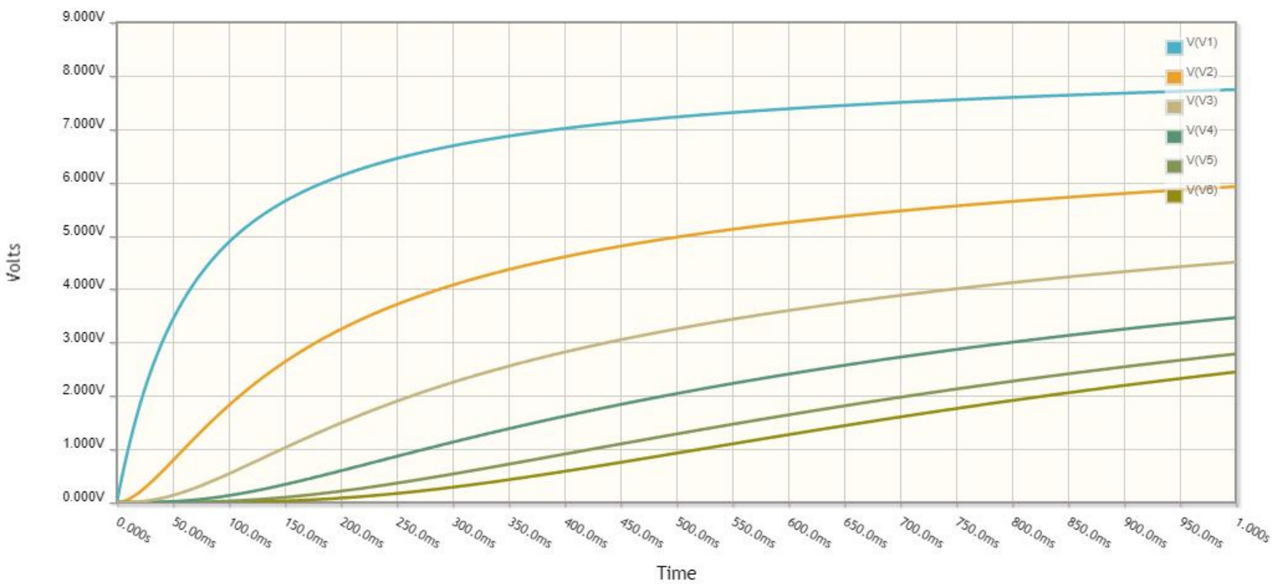
Code:

```
% 10-stage RC Filter

V = zeros(10,1);

V = zeros(10,1);

dV = zeros(10,1);
V0 = 10;
dt = 0.01;
t = 0;
i = 0;

y = [];

while(t < 4.99)


   dV(1)  = 11.111*V0   - 22.778*V(1) + 11.111*V(2);
   dV(2)  = 11.111*V(1) - 22.778*V(2) + 11.111*V(3);
   dV(3)  = 11.111*V(2) - 22.778*V(3) + 11.111*V(4);
   dV(4)  = 11.111*V(3) - 22.778*V(4) + 11.111*V(5);
   dV(5)  = 11.111*V(4) - 22.778*V(5) + 11.111*V(6);
   dV(6)  = 11.111*V(5) - 22.778*V(6) + 11.111*V(7);
   dV(7)  = 11.111*V(6) - 22.778*V(7) + 11.111*V(8);
   dV(8)  = 11.111*V(7) - 22.778*V(8) + 11.111*V(9);
   dV(9)  = 11.111*V(8) - 22.778*V(9) + 11.111*V(10);
   dV(10) = 11.111*V(9) - 11.667*V(10);

   V = V +  dV*dt;
   t = t + dt;

   y = [y ; V'];

   plot([0:10], [V0;V], '.-', t, 0, 'b+');
   xlim([0,10]);
   ylim([0,10]);
   pause(0.01);

end

pause(5);

t = [1:length(y)]' * dt;
plot(t,y);
xlabel('Time (seconds)');
ylabel('Volts');
```

7) Using CircuitLab, find the response of this circuit to a 10V step input. *note: It's OK if you only build this circuit to 3 nodes...*

Running with six nodes:





The response is essentially what we got with Matlab (only with six nodes instead of 10)

Natural Response: Eigenvectors and Eigenvalues

8) Assume V0 = 0V. Determine the initial conditions of V1..V10 so that

- The maximum voltage is 10V and
- The voltages go to zero as slow as possible
- The voltages go to zero as fast as possible.

Simulate the response for these initial conditions in Matlab.

This is an eigenvalue / eigenvector probem. Expressing the dynamics in matrix form:

```
>> A = zeros(10,10);
>> for i=1:9
A(i,i) = -22.778;
A(i+1,i) = 11.111;
A(i,i+1) = 11.111;
end
>> A(10,10) = -11.667

A =

  -22.7780   11.1110         0         0         0         0         0         0         0         0
   11.1110  -22.7780   11.1110         0         0         0         0         0         0         0
         0   11.1110  -22.7780   11.1110         0         0         0         0         0         0
         0         0   11.1110  -22.7780   11.1110         0         0         0         0         0
         0         0         0   11.1110  -22.7780   11.1110         0         0         0         0
         0         0         0         0   11.1110  -22.7780   11.1110         0         0         0
         0         0         0         0         0   11.1110  -22.7780   11.1110         0         0
         0         0         0         0         0         0   11.1110  -22.7780   11.1110         0
         0         0         0         0         0         0         0   11.1110  -22.7780   11.1110
         0         0         0         0         0         0         0         0   11.1110  -11.6670

>> [M,V] = eig(A)

M =

  fast mode                                                                                 slow mode
   -0.1286   -0.2459    0.3412    0.4063    0.4352    0.4255    0.3780    0.2969   -0.1894    0.0650
    0.2459    0.4063   -0.4255   -0.2969   -0.0650    0.1894    0.3780    0.4352   -0.3412    0.1286
   -0.3412   -0.4255    0.1894   -0.1894   -0.4255   -0.3412    0.0000    0.3412   -0.4255    0.1894
    0.4063    0.2969    0.1894    0.4352    0.1286   -0.3412   -0.3780    0.0650   -0.4255    0.2459
   -0.4352   -0.0650   -0.4255   -0.1286    0.4063    0.1894   -0.3780   -0.2459   -0.3412    0.2969
    0.4255   -0.1894    0.3412   -0.3412   -0.1894    0.4255   -0.0000   -0.4255   -0.1894    0.3412
   -0.3780    0.3780    0.0000    0.3780   -0.3780   -0.0000    0.3780   -0.3780    0.0000    0.3780
    0.2969   -0.4352   -0.3412    0.0650    0.2459   -0.4255    0.3780   -0.1286    0.1894    0.4063
   -0.1894    0.3412    0.4255   -0.4255    0.3412   -0.1894    0.0000    0.1894    0.3412    0.4255
    0.0650   -0.1286   -0.1894    0.2459   -0.2969    0.3412   -0.3780    0.4063    0.4255    0.4352

>> eig(A)'

  -44.0127  -41.1387  -36.6332  -30.8966  -24.4387  -17.8331  -11.6670   -6.4881   -2.7567   -0.8042
```

Eigenvalues tell you how the system behaves

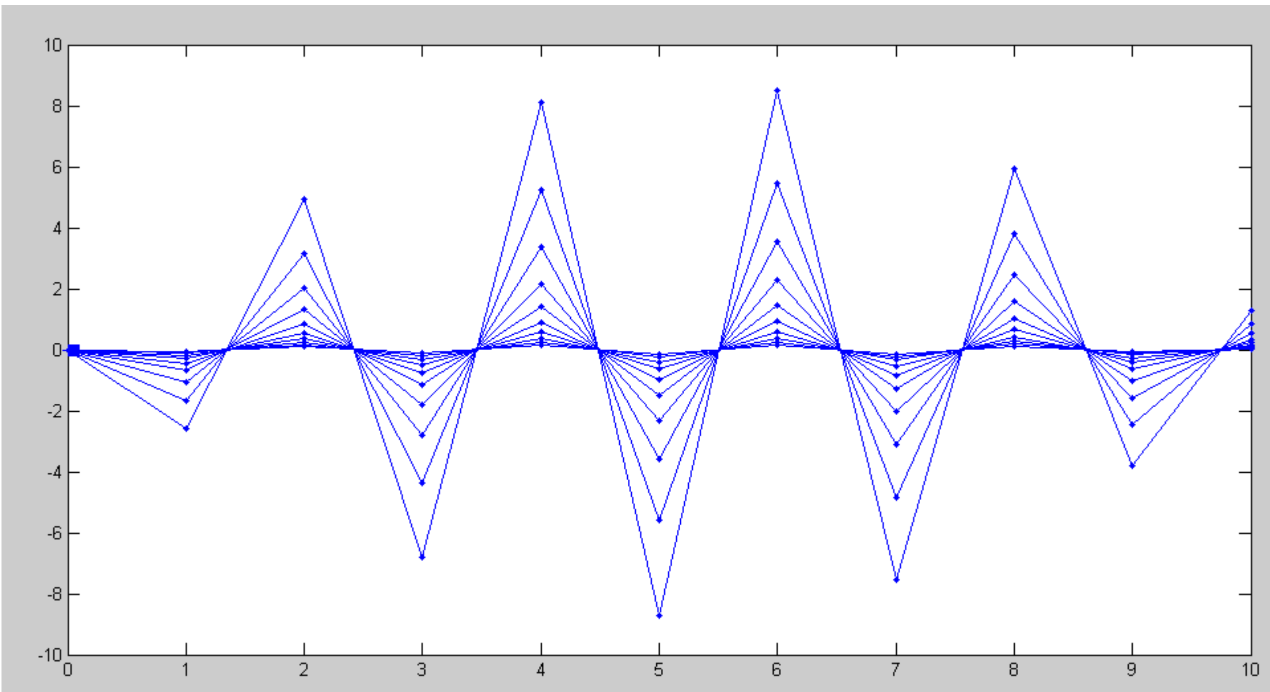
- Something decays as exp(-44.0127t)   *fast mode*
- Something decays as exp(-0.8042t)   *slow mode*

Eigenvectors tell you what behaves that way

Fast Mode: Make the initial condition proprtional to the fast eigenvector

Code:

```
% 10-stage RC Filter

V = 20 * M(:,1);

dV = zeros(10,1);
V0 = 0;
dt = 0.001;
t = 0;
i = 0;

y = [];

while(t < 0.99)
(etc)
```

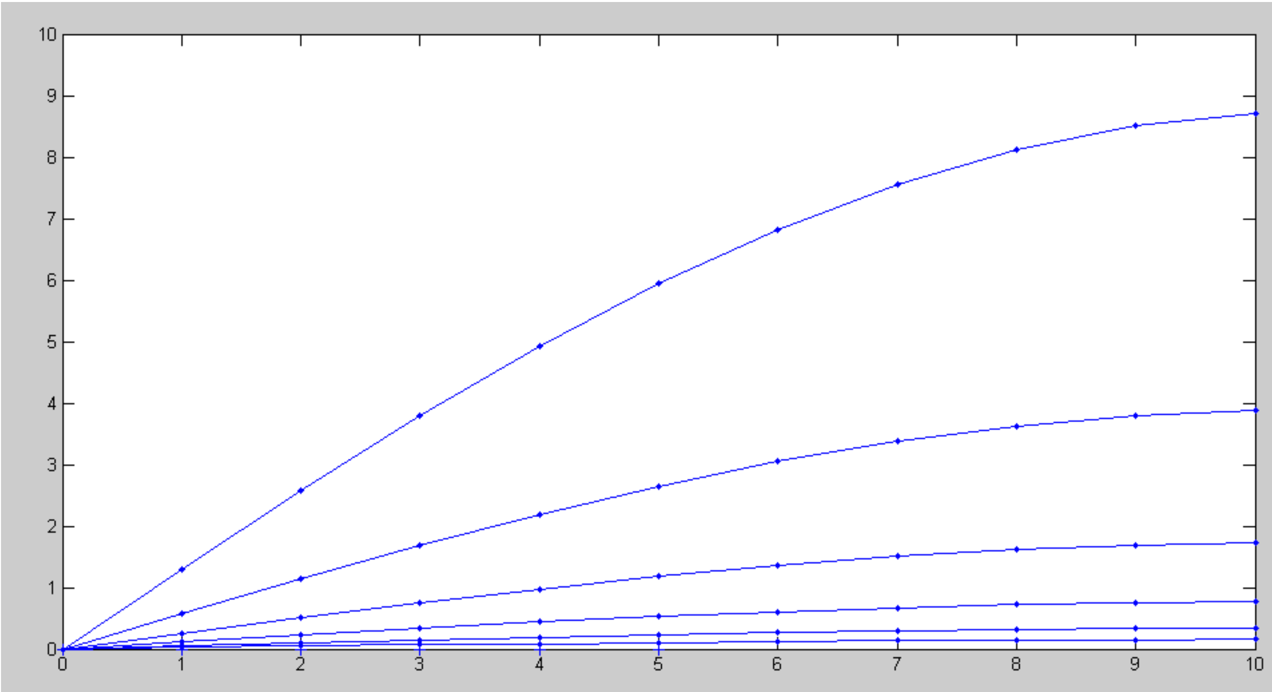


Fast Mode: Voltage plotted every 10ms

Note:

- The shape remains the same (the eigenvector)
- The amplitude decays quickly (the fast eigenvalue)

Slow Mode:



Voltages plotted every 1.000 second.

Note that the shape remains the same

- the eigenvector

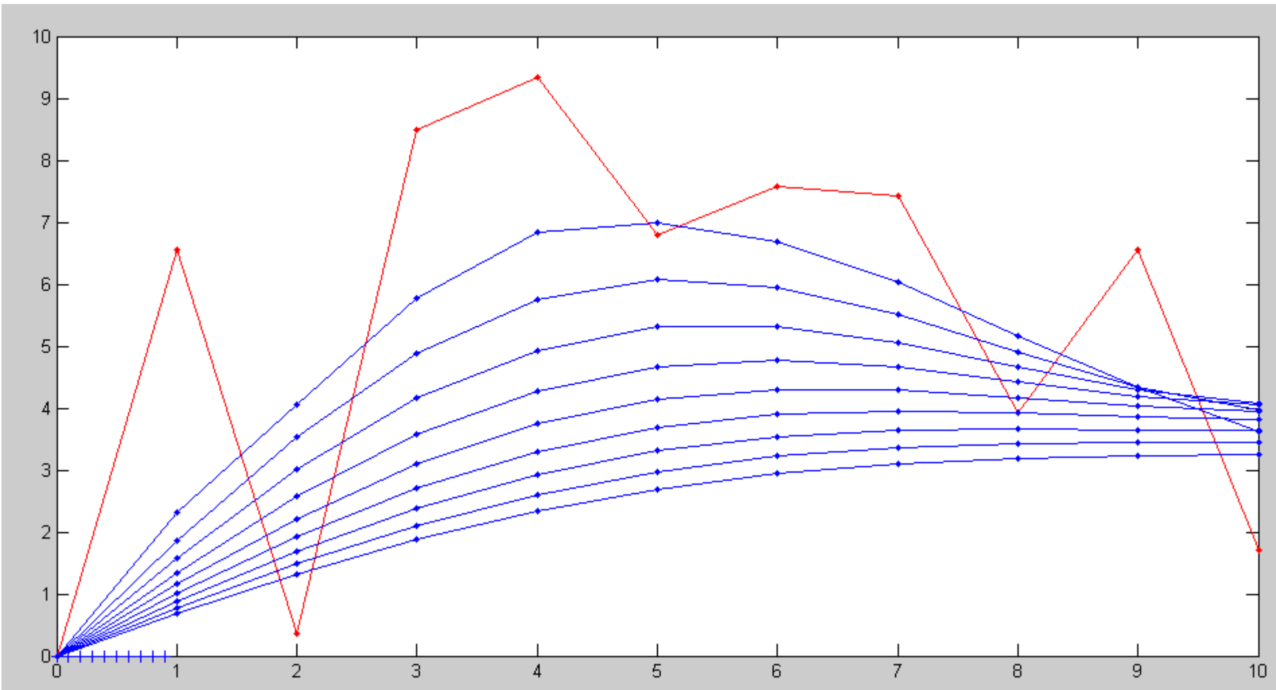
The amplitude decays slowly

- the eigenvalue
-

9) Assume Vin = 0V. Pick random voltages for V1 .. V10 in the range of (0V, 10V):

```
V = 10 * rand(10,1)
```

Plot the votlages at t = 2. Which eigenvector does it look like?



Voltages plotted every 100ms with a random initial condition (red)

Note:

- The initial voltage includes all ten eigenvectors
- The fast modes decay quickly
- Leaving the slow mode

After 2 seconds, what you see is the slow eigenvector