MIMO LQG Control with Servo Compensators

Work In Progress

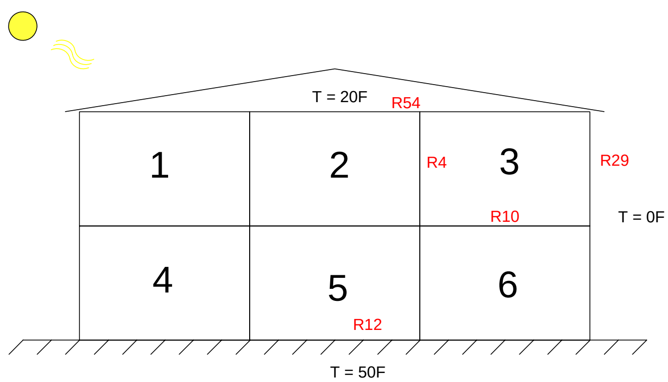
Multi-Input, Multi-Output Systems

With one input, you can stabilize a system with full-state feedback and you can control one output. With N inputs, however, you can control N outputs.

For example, consider the problem of controlling the temperature in a 6-apartment building, where each room has a heating unit (U1..U6) and there is an external disturbance from the sun shining on rooms 1 and 4 (adding one unit of heat).

This can be modeled as a 6-element RC filter, where

- R is the thermal resistance between rooms (or outside the building),
- C is the thermal capacitance of each room, and
- The input is the heat added to the room through U or from solar heating:



6-Room Apartment Buiilding

Case 1: Single Input System. Assume all heaters are tied together so that they all output the same amount of heat. In this case, the dynamics become:

$$sX = \begin{bmatrix} -0.403 & 0.25 & 0 & 0.1 & 0 & 0 \\ 0.25 & -0.683 & 0.25 & 0 & 0.1 & 0 \\ 0 & 0.25 & -0.468 & 0 & 0 & 0.1 \\ 0.1 & 0 & 0 & -0.468 & 0.25 & 0 \\ 0 & 0.1 & 0 & 0.25 & -0.683 & 0.25 \\ 0 & 0 & 0.1 & 0 & 0.25 & -0.468 \end{bmatrix} X + \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} U + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} 10W$$

With a single input, you can control a single output. Assume the thermostat is placed in room #2.

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} X$$

Add a servo-compensator to control the temperature of room #2.

$$\begin{bmatrix} sX \\ sZ \end{bmatrix} = \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix} \begin{bmatrix} X \\ Z \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} U + \begin{bmatrix} 0 \\ -1 \end{bmatrix} R + \begin{bmatrix} B_d \\ 0 \end{bmatrix} d$$

Since you are controlling one ouput (the average of all six), you only need one input.

Option 1: Apply the same amount of heat to all six heaters (equal heating bill for all tenants)

Input the system dynamics:

```
>> A = [-0.403,0.25,0,0.1,0,0;
         0.25,-0.683,0.25,0,0.1,0;
           0,0.25,-0.468,0,0,0.1;
         0.1, 0 0 -0.468 0.25 0;
         0 0.1 0 0.25 -0.683 0.25;
         0 0 0.1 0 0.25 -0.468];

>> B = [1 1 1 1 1 1]';

>> C = [0 1 0 0 0 0]
```

Add a servo-compensator to control the temperature of room #2:

```
>> A7 = [A, zeros(6,1);  C, 0]

   -0.4030    0.2500         0    0.1000         0         0 :         0
    0.2500   -0.6830    0.2500         0    0.1000         0 :         0
         0    0.2500   -0.4680         0         0    0.1000 :         0
    0.1000         0         0   -0.4680    0.2500         0 :         0
         0    0.1000         0    0.2500   -0.6830    0.2500 :         0
         0         0    0.1000         0    0.2500   -0.4680 :         0
 - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - -
         0    1.0000         0         0         0         0 :         0
```

Design a feedback control law. Assume all rooms apply the same heat:

```
>> B7 = [B; 0]

      1
      1
      1
      1
      1
      1
   - - - -
      0

>> Kx = lqr(A7, B7, diag([0,0,0,0,0,0,1e3]), 1)

    0.2434    7.2839    0.2428    0.0019    0.0959    0.0020   31.6228
```

```
>> eig(A7 - B7*Kx)

  -3.9765 + 3.9762i
  -3.9765 - 3.9762i
  -1.0285
  -0.8545
  -0.5519
  -0.3057
  -0.3495
```

If R is set to 70F, the temperature in each room (with no disturbance) is

```
>> Br = [0;0;0;0;0;0;-1]

     0
     0
     0
     0
     0
     0
    -1

>> DC = -inv(A7 - B7*Kx)*(Br*70)

   75.2178
   70.0000      Room #2 is controlled to 70F as desired
   64.2711
   64.9276
   66.1773
   62.5886
  -17.6046

>> U = -K7*DC

    6.3200
```

If the sun is shining on rooms 1 and 4 with 10 Watt of heat, then

```
>> -inv(A7 - B7*Kx)*(Br*70 + Bd*10)

   96.8808
   70.0000      Room #2 is still 70F in spite of the disturbance
   55.6075
   83.6270
   65.0763
   53.4404
  -17.6028

>> U = -K7*DC

U =

    3.1803
```

Two Inputs:

With a single input, you can only control one output. If you have two inputs, you can control two outputs.

Assume you separate the inputs into 1st and 2nd floor:

$$sX = \begin{bmatrix} -0.403 & 0.25 & 0 & 0.1 & 0 & 0 \\ 0.25 & -0.683 & 0.25 & 0 & 0.1 & 0 \\ 0 & 0.25 & -0.468 & 0 & 0 & 0.1 \\ 0.1 & 0 & 0 & -0.468 & 0.25 & 0 \\ 0 & 0.1 & 0 & 0.25 & -0.683 & 0.25 \\ 0 & 0 & 0.1 & 0 & 0.25 & -0.468 \end{bmatrix} X + \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} U + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} 10W$$

With two inputs, you can control two outputs. Let those outputs be room 2 and 4:

$$Y = \begin{bmatrix} x_2 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} X$$

Design a servo for each output:

$$\begin{bmatrix} sX \\ sz_1 \\ sz_2 \end{bmatrix} = \begin{bmatrix} A & 0 & 0 \\ C_1 & 0 & 0 \\ C_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} X \\ z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} B \\ 0 \\ 0 \end{bmatrix} U + \begin{bmatrix} 0 & 0 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} R + \begin{bmatrix} B_d \\ 0 \\ 0 \end{bmatrix} d$$

Find a feedback control law: use LQR with Q = 100I, R = 1

```
>> Az = [0,0;0,0]

Az =

     0     0
     0     0

>> C

C =

     0     1     0     0     0     0

>> C = [0,1,0,0,0,0;0,0,0,1,0,0]

C =

     0     1     0     0     0     0
     0     0     0     1     0     0
```

```
>> B = [1,1,1,0,0,0;0,0,0,1,1,1]'

     1     0
     1     0
     1     0
     0     1
     0     1
     0     1

>> C = [0,1,0,0,0,0;0,0,0,1,0,0]

     0     1     0     0     0     0
     0     0     0     1     0     0

>> A8 = [A, zeros(6,2) ; C, Az]


   -0.4030    0.2500         0    0.1000         0         0 :         0         0
    0.2500   -0.6830    0.2500         0    0.1000         0 :         0         0
         0    0.2500   -0.4680         0         0    0.1000 :         0         0
    0.1000         0         0   -0.4680    0.2500         0 :         0         0
         0    0.1000         0    0.2500   -0.6830    0.2500 :         0         0
         0         0    0.1000         0    0.2500   -0.4680 :         0         0
 - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - -
         0    1.0000         0         0         0         0 :         0         0
         0         0         0    1.0000         0         0 :         0         0


>> B8 = [B; zeros(2,2)]


     1     0
     1     0
     1     0
     0     1
     0     1
     0     1
     0     0
     0     0

>> B8r = [zeros(6,2); -eye(2,2)]

     0     0
     0     0
     0     0
     0     0
     0     0
     0     0
    -1     0
     0    -1
```

```
>> Bd = [1;0;0;1;0;0;0;0]

     1          sun shines on apt #1
     0
     0
     1          and #4
     0
     0
     0
     0


>> Kx = lqr(A8, B8, diag([0,0,0,0,0,0,1e3,1e3]), diag([1,1]))

                      Kx                                    :        Kz
    0.2436    7.2856    0.2431    0.0002    0.0957    0.0020 :  31.6228   -0.0001
    0.0980    0.0004   -0.0005    7.4939    0.2432    0.0024 :   0.0001   31.6228


>> eig(A8 - B8*Kx)

  -0.3492
  -0.5522
  -1.0354
  -0.8352
  -3.9770 + 3.9757i
  -3.9770 - 3.9757i
  -3.9794 + 3.9733i
  -3.9794 - 3.9733i
```

The steady-state response on a cloudy day (no disturbance)

```
>> DC = -inv(A8 - B8*Kx)*Br*[70; 80]

   75.1517
   70.0000            apt #2 is tracking its setpoint
   64.2146
   80.0000            apt #4 is tracking its setpoint
   81.8227
   77.6630
  -17.6043
  -20.1253

 U = -Kx*DC

    4.7861            2nd floor is adding 4.78W of heat in each apt
    9.4692            1st floor is adding 9.46W of heat in each apt
```

The steady-state response on a sunny day (with the disturbance)

```
>> DC = -inv(A8 - B8*Kx)*(Br*[70; 80] + Bd*10)

   96.8968
   70.0000      apt #2 remains 70F
   55.6211
   80.0000      apt #4 remains 80F
   61.3114
   49.8129
  -17.6028
  -19.8102
```

The servo compensators are doing their job: they track a constant set point and reject constant disturbances.

The input in this case is:

```
>> -Kx*DC

   3.5494       Watts of heat for 2nd floor apts
   2.4225       Watts of heat for 1st floor apts
```

Six Inputs:

With six inputs, you can control six outputs. Assume each room controls its own heater:

```
>> A

   -0.4030    0.2500         0    0.1000         0         0
    0.2500   -0.6830    0.2500         0    0.1000         0
         0    0.2500   -0.4680         0         0    0.1000
    0.1000         0         0   -0.4680    0.2500         0
         0    0.1000         0    0.2500   -0.6830    0.2500
         0         0    0.1000         0    0.2500   -0.4680

>> B = eye(6,6)

     1     0     0     0     0     0
     0     1     0     0     0     0
     0     0     1     0     0     0
     0     0     0     1     0     0
     0     0     0     0     1     0
     0     0     0     0     0     1

>> C = eye(6,6)

     1     0     0     0     0     0
     0     1     0     0     0     0
     0     0     1     0     0     0
     0     0     0     1     0     0
     0     0     0     0     1     0
     0     0     0     0     0     1
```

```
>> A12 = [A, zeros(6,6) ; C, zeros(6,6) ]

   -0.4030    0.2500         0    0.1000         0         0         0         0         0         0         0         0
    0.2500   -0.6830    0.2500         0    0.1000         0         0         0         0         0         0         0
         0    0.2500   -0.4680         0         0    0.1000         0         0         0         0         0         0
    0.1000         0         0   -0.4680    0.2500         0         0         0         0         0         0         0
         0    0.1000         0    0.2500   -0.6830    0.2500         0         0         0         0         0         0
         0         0    0.1000         0    0.2500   -0.4680         0         0         0         0         0         0
    1.0000         0         0         0         0         0         0         0         0         0         0         0
         0    1.0000         0         0         0         0         0         0         0         0         0         0
         0         0    1.0000         0         0         0         0         0         0         0         0         0
         0         0         0    1.0000         0         0         0         0         0         0         0         0
         0         0         0         0    1.0000         0         0         0         0         0         0         0
         0         0         0         0         0    1.0000         0         0         0         0         0         0

>> B12 = [B; zeros(6,6)]

     1     0     0     0     0     0
     0     1     0     0     0     0
     0     0     1     0     0     0
     0     0     0     1     0     0
     0     0     0     0     1     0
     0     0     0     0     0     1
     0     0     0     0     0     0
     0     0     0     0     0     0
     0     0     0     0     0     0
     0     0     0     0     0     0
     0     0     0     0     0     0
     0     0     0     0     0     0

>> Br = [zeros(6,6); -eye(6,6)]

     0     0     0     0     0     0
     0     0     0     0     0     0
     0     0     0     0     0     0
     0     0     0     0     0     0
     0     0     0     0     0     0
     0     0     0     0     0     0
    -1     0     0     0     0     0
     0    -1     0     0     0     0
     0     0    -1     0     0     0
     0     0     0    -1     0     0
     0     0     0     0    -1     0
     0     0     0     0     0    -1

>> Bd = [1 0 0 1 0 0 0 0 0 0 0 0];


>> Kx =  lqr(A12, B12, diag([0,0,0,0,0,0,1,1,1,1,1,1])*1000,eye(6,6))

    7.5644    0.2330    0.0039    0.0945    0.0031    0.0000   31.6228    0.0000   -0.0000   -0.0000   -0.0000    0.0000
    0.2330    7.3074    0.2320    0.0031    0.0915    0.0031    0.0000   31.6228   -0.0000    0.0000    0.0000    0.0000
    0.0039    0.2320    7.5030    0.0000    0.0031    0.0941   -0.0000   -0.0000   31.6228   -0.0000    0.0000   -0.0000
    0.0945    0.0031    0.0000    7.5030    0.2320    0.0039    0.0000    0.0000    0.0000   31.6228   -0.0000    0.0000
    0.0031    0.0915    0.0031    0.2320    7.3074    0.2320   -0.0000    0.0000    0.0000   -0.0000   31.6228   -0.0000
    0.0000    0.0031    0.0941    0.0039    0.2320    7.5030    0.0000    0.0000    0.0000   -0.0000    0.0000   31.6228
```

```
>> eig(A12 - B12*Kx)

  -4.0102 + 3.9422i
  -4.0102 - 3.9422i
  -3.9985 + 3.9541i
  -3.9985 - 3.9541i
  -3.9860 + 3.9667i
  -3.9860 - 3.9667i
  -3.9766 + 3.9761i
  -3.9766 - 3.9761i
  -3.9790 + 3.9737i
  -3.9790 - 3.9737i
  -3.9803 + 3.9724i
  -3.9803 - 3.9724i
```

Steady-state response on a cloudy day (no disturbance)

```
>> DC = -inv(A12 - B12*Kx)*(Br*[60 65 70 75 80 85]')

   60.0000      Apt #1 tracks its set point
   65.0000
   70.0000
   75.0000
   80.0000
   85.0000      Apt #6 tracks its set point
  -15.0858
  -16.3461
  -17.6070
  -18.8658
  -20.1184
  -21.3826

>> U = -Kx*DC

    0.4300      Watts of heat at apt #1
    3.8950
    8.0100
    9.1000
    8.1400
   12.7800      Watts of heat at apt #6
```

Stead-State Response on a Sunny Day (with a disturbance)

```
>> DC = -inv(A12 - B12*Kx)*(Br*[60 65 70 75 80 85]' + Bd*10)

   60.0000       Apt #1 still tracks its set point
   65.0000
   70.0000
   75.0000
   80.0000
   85.0000       Apt #6 still tracks its set point
  -14.7696
  -16.3461
  -17.6070
  -18.5496
  -20.1184
  -21.3826


>> U = -Kx*DC

   -9.5700             Watts of heat added for apt #1 (negative means cooling)
    3.8950
    8.0100
   -0.9000
    8.1400
   12.7800             Watts of heat added for apt #6

>>
```