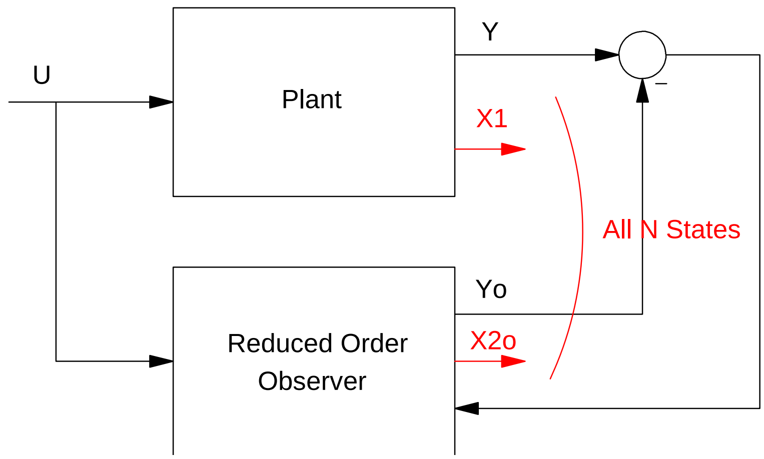
Reduced-Order Observers

The purpose of observers is to estimate the states that are not measured directly. If a few of these states *are* measured, however, they do not need to be estimated. Likewise, it would seem to make sense to use the actual measurements when available and only use the state estimates when the corresponding state is not measured.



If you can measure m states, then you only need to estimate the remaining N-m states

Suppose you only need to measure a few of the states. Can you build a state-estimator that estimates only the states that are *not* measured?

The answer is 'yes' and the result is termed a *reduced-order observer*.

Derivation

Assume you have a dynamic system which is observable:

$$sX = AX + BU$$

$$Y = CX$$

Assume that some of the states are measured directly. Separate X into the states which are directly measured and those which are not:

$$s\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}U$$

$$Y = \begin{bmatrix} C_1 & 0 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

where C_1 is invertable (i.e. X1 is measured directly).

A full-order observer is

$$s\begin{bmatrix} \hat{X}_1 \\ \hat{X}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} \hat{X}_1 \\ \hat{X}_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} U + \begin{bmatrix} H_1 \\ H_2 \end{bmatrix}\left(Y - C_1\hat{X}\right)$$

Since X1 is measured directly, let

$$\hat{X}_1 = C_1^{-1}Y$$

Then, the full-order observer becomes

$$s\hat{X}_2 = A_{21}C_1^{-1}Y + A_{22}\hat{X}_2 + B_2U$$

Let

$$\hat{X}_2 = LY + Z$$

where Z is from the system

$$sZ = FZ + GY + HU$$

Then

$$sE_2 = sX_2 - s\hat{X}_2 = A_{21}X_1 + A_{22}X_2 + B_2U - LsY - sX$$

$$sE_2 = A_{21}X_1 + A_{22}X_2 + B_2U - L(C_1(A_{11}X_1 + A_{12}X_2 + B_1U)) - FZ - GY - HU$$

but

$$Z = \hat{X}_2 - LY = X_2 - E_2 - LY = X_2 - E_2 - LC_1X_1$$

so

$$sE_2 = A_{21}X_1 + A_{22}X_2 + B_2U - L(C_1(A_{11}X_1 + A_{12}X_2 + B_1U)) - F(X_2 - E_2 - LC_1X_1) - GY - HU$$

Grouping terms

$$sE_2 = (A_{21} - LC_1A_{11} - GC_1 + FLC_1)X_1 + (A_{22} - LC_1A_{12} - F)X_2 + (B_2 - LC_1B_1 - H)U$$

In order for the error to be driven to zero,

$$GC_1 = A_{21} - LC_1A_{11} + FLC_1$$

$$F = A_{22} - LC_1A_{12}$$

$$H = B_2 - LC_1B_1$$

Then, the error dynamics beocme

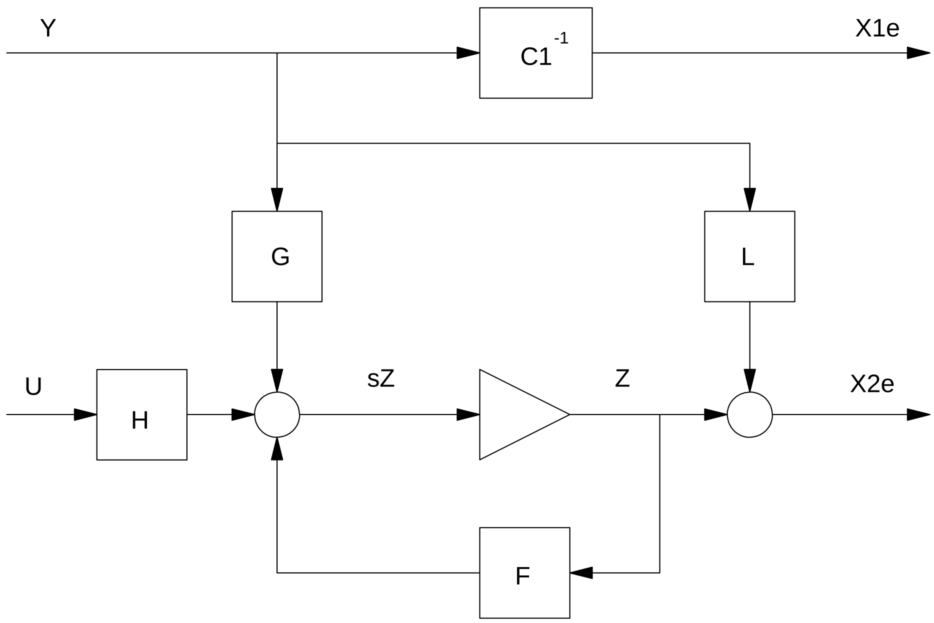
$$sE_2 = FE_2$$

If F is chosen so that it is negative definite, the error will be driven to zero. To do this, you can use pole-placement to find L to place the poles of:

$$F = A_{22} - L(C_1 A_{12})$$

```
-->A11 = -1;
-->A12 = [1,0,0];
-->A21 = [1;0;0];
-->A22 = [-2,1,0;1,-2,1;0,1,-2];


-->B1 = 0;
-->B2 = [0;0;1];

-->C1 = 1;
-->C2 = [0,0,0];
```

```
-->L = ppl(A22',(C1*A12)',[-3,-4,-5])'

    6.
    13.
    12.


-->F = A22 - L*C1*A12
 F  =

  - 8.      1.     0.
  - 12.  - 2.     1.
  - 12.     1.  - 2.

-->H = B2-L*C1*B1
 H  =

    0.
    0.
    1.

-->G = inv(C1) * (A21 - L*C1*A11 + F*L*C1)
 G  =

  - 28.
  - 73.
  - 71.
```



The augmented system is then

$$s\begin{bmatrix} X \\ Z \end{bmatrix} = \begin{bmatrix} A & 0 \\ GC & F \end{bmatrix}\begin{bmatrix} X \\ Z \end{bmatrix} + \begin{bmatrix} B \\ H \end{bmatrix}U$$

$$\begin{bmatrix} \hat{X}_1 \\ \hat{X}_2 \end{bmatrix} = \begin{bmatrix} C_1^{-1}C & 0 \\ LC & I \end{bmatrix}\begin{bmatrix} X \\ Z \end{bmatrix}$$

Example:

Reduced-Order Observer for a heat equation. Let

$$sX = \begin{bmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}X + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}U$$

$$Y = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}X$$

Design a reduced-order observer to estimate states 1..3

Rewrite this as

$$sX = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix} X + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} U$$

$$Y = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} X$$

Separate into sections:

$$s\begin{bmatrix} X_1 \\ \cdots \\ X_2 \end{bmatrix} = \begin{bmatrix} -1 & \vdots & 1 & 0 & 0 \\ \cdots & & \cdots & \cdots & \cdots \\ 1 & \vdots & -2 & 1 & 0 \\ 0 & \vdots & 1 & -2 & 1 \\ 0 & \vdots & 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} X_1 \\ \cdots \\ X_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \cdots \\ 0 \\ 0 \\ 1 \end{bmatrix} U$$

$$Y = \begin{bmatrix} 1 & \vdots & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ \cdots \\ X_2 \end{bmatrix}$$

Pick L to stabilize F:

$$F = A_{22} - L(C_1 A_{12})$$

In Matlab

```
A11 = -1;
A12 = [1,0,0];
A21 = [1;0;0];
A22 = [-2,1,0;1,-2,1;0,1,-2];

B1 = 0;
B2 = [0;0;1];

C1 = 1;
C2 = [0,0,0];


L = ppl(A22',(C1*A12)',[-3,-4,-5])'

    6.
    13.
    12.
```

```
F = A22 - L*C1*A12

  - 8.      1.     0.
  - 12.   - 2.     1.
  - 12.     1.   - 2.

H = B2-L*C1*B1

    0.
    0.
    1.

G = inv(C1) * (A21 - L*C1*A11 + F*L*C1)

  - 28.
  - 73.
  - 71.

eig(F)

  - 5.
  - 4.
  - 3.
```

The system plus the reduced-order observer is then:

```
A7 = [A, zeros(4,3);G*C,F]

  - 1.      1.     0.     0. :   0.      0.     0.
    1.    - 2.     1.     0. :   0.      0.     0.
    0.      1.   - 2.     1. :   0.      0.     0.
    0.      0.     1.   - 2. :   0.      0.     0.
  - - - - - - - - - - - - :- - - - - - - - - - -
  - 28.     0.     0.     0. : - 8.      1.     0.
  - 73.     0.     0.     0. : - 12.   - 2.     1.
  - 71.     0.     0.     0. : - 12.     1.   - 2.


B7 = [B;H]

    0.
    0.
    0.
    1.
- - - -
    0.
    0.
    1.


Cx = [C,zeros(1,3);L*C,eye(3,3)]

    1.      0.     0.     0.     0.     0.     0.
    6.      0.     0.     0.     1.     0.     0.
    13.     0.     0.     0.     0.     1.     0.
    12.     0.     0.     0.     0.     0.     1.
```

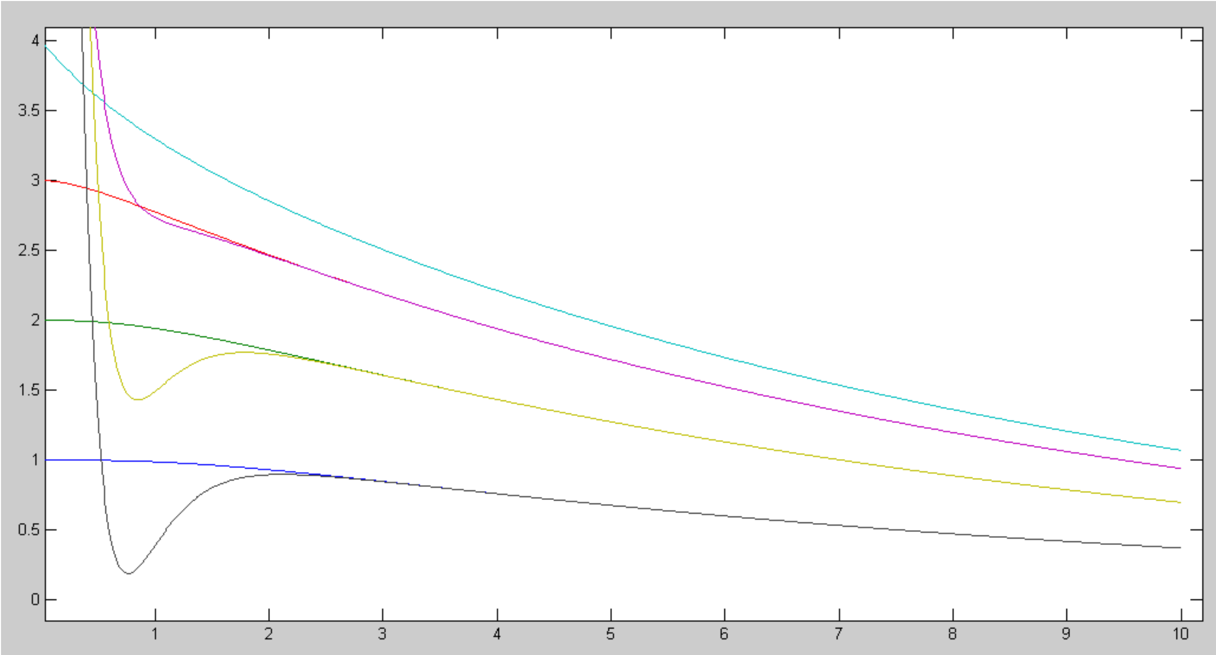
Plotting the response of the observer with the plant having an initial condition of {1, 2, 3, 4} is:

First, create the augmented system (plant plus reduced order observer)

```
>> A7 = [A, zeros(4,3);G*C,F]
>> B7 = [B;H]
>> Cx = [C,zeros(1,3);L*C,eye(3,3)]
```

To see if the observer states converge, plot all four plant states as well:

```
>> C7 = [eye(3,4), zeros(3,3);  Cx]

    1.0000         0         0         0         0         0         0
         0    1.0000         0         0         0         0         0
         0         0    1.0000         0         0         0         0
         0         0         0    1.0000         0         0         0
         0         0         0    6.0000    1.0000         0         0
         0         0         0   13.0000         0    1.0000         0
         0         0         0   12.0000         0         0    1.0000

>> X0 = [1;2;3;4;0;0;0]

     1         plant states
     2
     3
     4
   - - -
     0
     0         reduced-order observer states
     0

>> G7 = ss(A7, X0, C7, zeros(7,1));
>> y7 = impulse(G7,t);
>> plot(t,y7)
```



Response of a Reduced-Order Observer with the Plant Initial States Being {1, 2, 3, 4}

Note

- The observer states converge
- The settling time is about 2 seconds (observer poles were placed at -3, -4, -5)
- You get fewer states, which isn't that big of a deal since it will be implimented in software
- In exchange for a much more complicated design - which is a big deal.