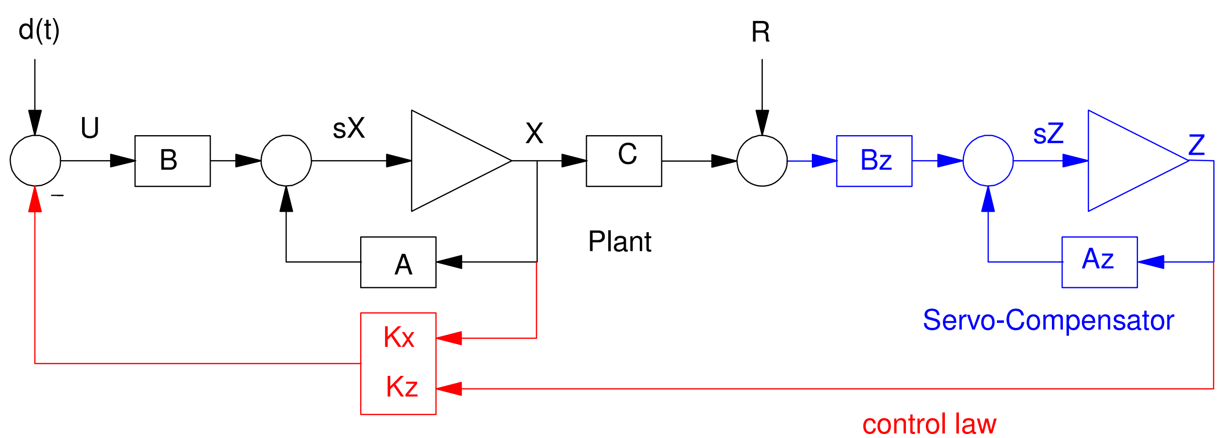
Servo-Compensators: General Design

From before,

- If you are trying to track a constant set-pount (R) and/or reject a constant disturbance (d), you add a servo compensator with poles at s = 0.
- If you are trying to track a sinusoidal set-point (R) at frequency ω and/or reject a sinusoidal disturbance at frequency ω, you add a servo compensator with poles at $s = \pm j\omega$.

Not surprisingly,

- If you are trying to track a set point with a spectra at $\{\ 0,\ \ j\omega,\ \ -j\omega\ \}$ and/or reject a disturbance with a spectra at $\{\ 0,\ \ j\omega,\ \ -j\omega\ \}$, you add a servo-compensator with poles at $\{\ 0,\ \ j\omega,\ \ -j\omega\ \}$.



Example: Let the plant be

$$sX = AX + BU$$

$$Y = CX$$

Define a servo-compensator

$$sZ = A_z Z + B_z$$

so that the eigenvalue of Az are

$$eig(A_z) = 0, \pm j\omega$$

Feed the servo-compensator with the difference between Y and the set point R

In state-space, the plant plus servo-compensator looks like the following:

$$s\begin{bmatrix} X \\ Z \end{bmatrix} = \begin{bmatrix} A & 0 \\ B_zC & A_z \end{bmatrix}\begin{bmatrix} X \\ Z \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix}U + \begin{bmatrix} 0 \\ -B_z \end{bmatrix}R$$

$$U = -\begin{bmatrix} K_x & K_z \end{bmatrix}\begin{bmatrix} X \\ Z \end{bmatrix}$$

or you can write this as

$$s\begin{bmatrix} X \\ Z \end{bmatrix} = \begin{bmatrix} A - BK_x & -BK_z \\ A_z & B_zC \end{bmatrix}\begin{bmatrix} X \\ Z \end{bmatrix} + \begin{bmatrix} 0 \\ -B_z \end{bmatrix}R$$

Example:

Assume a 4th-order heat equation:

$$sX = \begin{bmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}X + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}U + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}d(t)$$

$$Y = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}X$$

Design a feedback control law for the following system so that

- The 2% settling time is 13 seconds,
- There is no overshoot for a step input,
- Y tracks a constant setpoint (R = 1), and
- Y rejects a sinusoidal disturbance at 1 rad/sec

$$d(t) = \sin(t)$$

Note: This also works for any combination of constant + 1 rad/sec set point (R) and disturbance (d)

$$R(t) = a_1 + b_1\cos(t) + c_1\sin(t)$$

$$d(t) = a_2 + b_2\cos(t) + c_2\sin(t)$$

Step 1: Add a servo compensator which is controllable and has poles at $\{0, j, -j\}$

$$sZ = \left[\begin{array}{cc:c} 0 & 1 & 0 \\ -1 & 0 & 0 \\ \cdots & \cdots & \cdots \\ 0 & 0 & 0 \end{array}\right]Z + \begin{bmatrix} 1 \\ 1 \\ \cdots \\ 1 \end{bmatrix}(R - Y)$$

Step 2: Create the augmented system: plant + servo compensator

$$s\begin{bmatrix} X \\ Z \end{bmatrix} = \begin{bmatrix} A & 0 \\ B_z C & A_z \end{bmatrix}\begin{bmatrix} X \\ Z \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} U + \begin{bmatrix} 0 \\ -B_z \end{bmatrix} R$$

$$s\begin{bmatrix} X \\ \cdots \\ Z \end{bmatrix} = \left|\begin{array}{cccc:ccc} -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right| \begin{bmatrix} X \\ \cdots \\ Z \end{bmatrix} + \left|\begin{array}{c} 1 \\ 0 \\ 0 \\ 0 \\ \cdots \\ 0 \\ 0 \\ 0 \end{array}\right| U + \left|\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ \cdots \\ -1 \\ -1 \\ -1 \end{array}\right| R$$

Design a full-state feedback control law to meet the design specs. Somwhat arbitrarilly, place the closed-loop poles at {-1, -2, -2.2, -2.3, -2.4, -0.3+j, -0.3-j} using Bass Gura

In Matlab:

```
A = [-2,1,0,0;1,-2,1,0;0,1,-2,1;0,0,1,-1]

    -2      1      0      0
     1     -2      1      0
     0      1     -2      1
     0      0      1     -1

B = [1;0;0;0]

     1
     0
     0
     0

C = [0,0,0,1]

     0      0      0      1

Az = [0,1,0;-1,0,0;0,0,0]

     0      1      0
    -1      0      0
     0      0      0

Bz = [1;1;1]

     1
     1
     1
```

```
A7 = [A, zeros(4,3) ; Bz*C, Az]

    -2      1      0      0   :   0      0      0
     1     -2      1      0   :   0      0      0
     0      1     -2      1   :   0      0      0
     0      0      1     -1   :   0      0      0
    ------------------------------------------
     0      0      0      1   :   0      1      0
     0      0      0      1   :  -1      0      0
     0      0      0      1   :   0      0      0

B7u = [B ; zeros(3,1)]

     1
     0
     0
     0
 - - - - -
     0
     0
     0


K7 = ppl(A7, B7u, [-1, -2, -2.2, -2.3, -2.4, -0.3+j, -0.3-j])

    3.5000   12.0900   29.6810   63.4358    0.5236   21.3719   26.4739
```

This gives

```
Kx = [ 3.5000   12.0900   29.6810   63.4358 ]
Kz = [ 0.5236   21.3719   26.4739 ]
```

To simulate the system, you need to add a constant and / or sinusoidal disturbance and set point. These have poles at {0, +j, -j} as well:

$$sX_r = A_r X_r$$

$$sX_r = \begin{bmatrix} 0 & 1 & \vdots & 0 \\ -1 & 0 & \vdots & 0 \\ \cdots & \cdots & \vdots & \cdots \\ 0 & 0 & \vdots & 0 \end{bmatrix} X_r$$

$$R = C_r X_r$$

$$d = C_d X_r$$

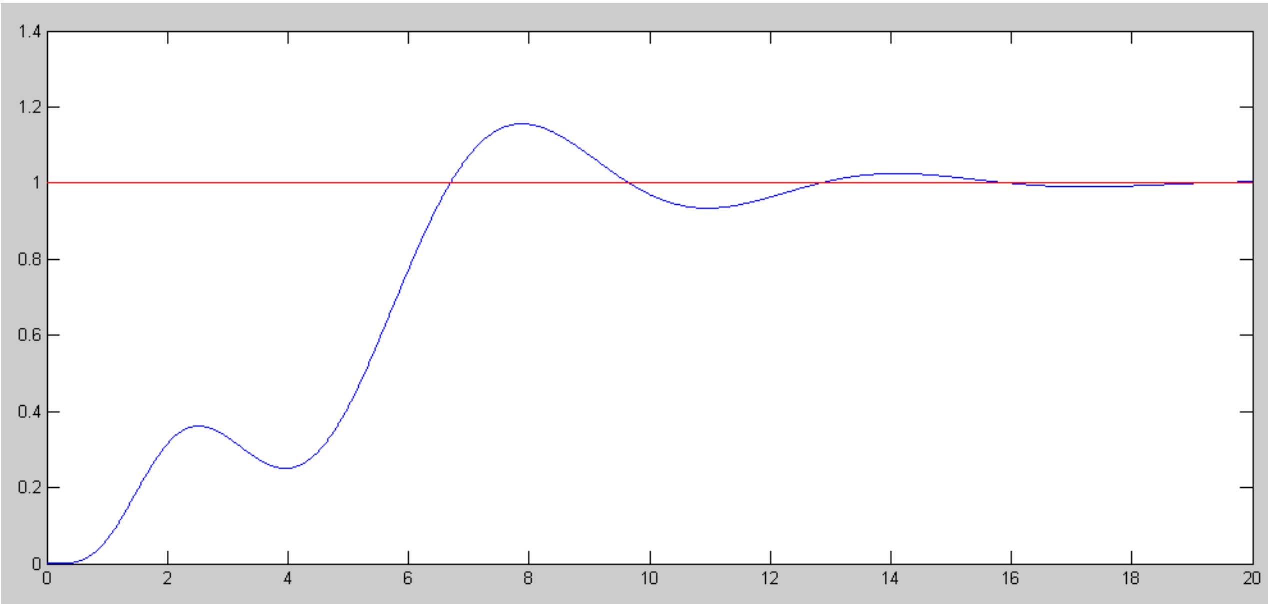
The initial condition determines the amplitude and phase shift of the sinusoid (first two states) and the constant (3rd state). Cr and Cd determine what the set point and disturbance are.

Using the Step3 command

```
function [ y ] = step3( A, B, C, D, t, X0, U )
```

The servo compensator can track a constant set point:

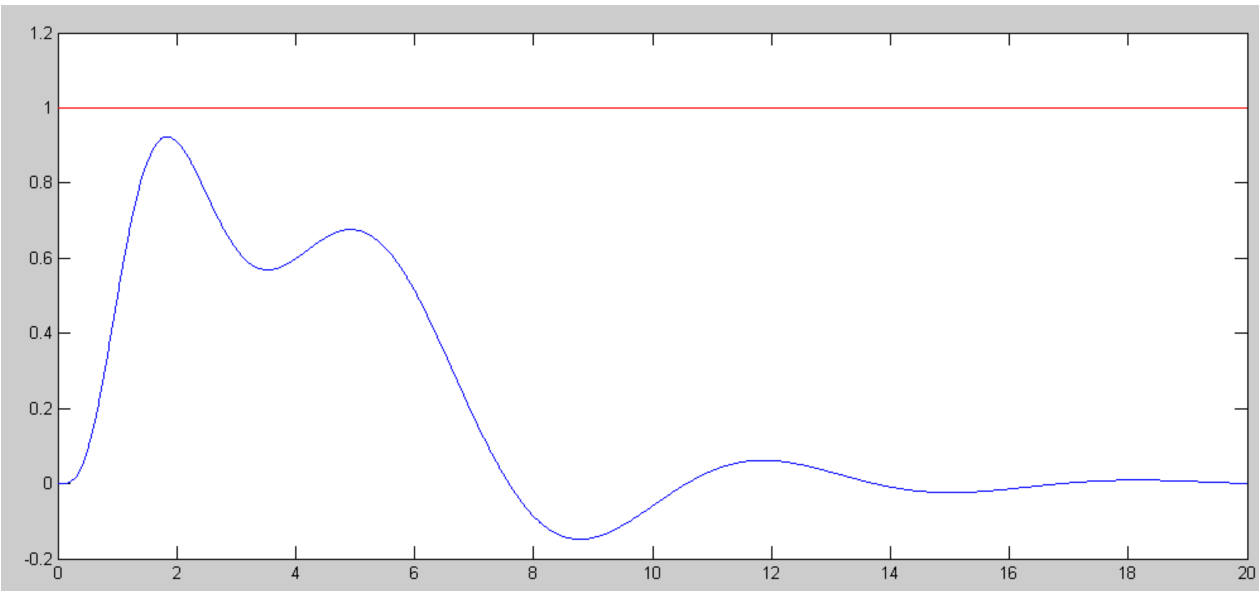
```
B7r = [0*B ; -Bz]

     0
     0
     0
     0
    -1
    -1
    -1

X0 = zeros(7,1);
t = [0:0.01:20]';
R = 0*t + 1;
y = step3(A7-B7u*K7, B7r, C7, 0, t, X0, R);
plot(t,R,,'r',t,y,'b')
```



Step Response with respect to R. The output tracks a constant set point

The servo compensator also rejects a constant disturbance:

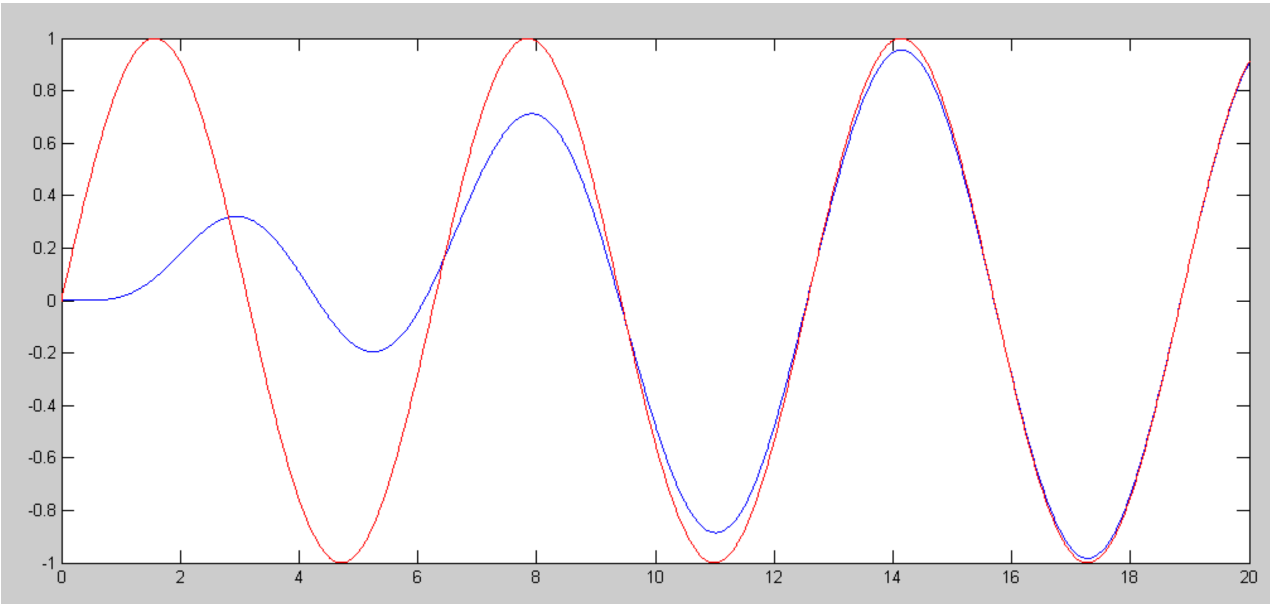
```
B7u = [B; 0 * Bz]

     1
     0
     0
     0
     0
     0
     0

y = step3(A7-B7u*K7, B7u*100, C7, 0, t, X0, R);
plot(t,R,t,y)
```



Response to a step disturbance (d = 100). The servo compensator rejects a constant disturbance.

The servo compensator can track a 1 rad/sec sine wave:

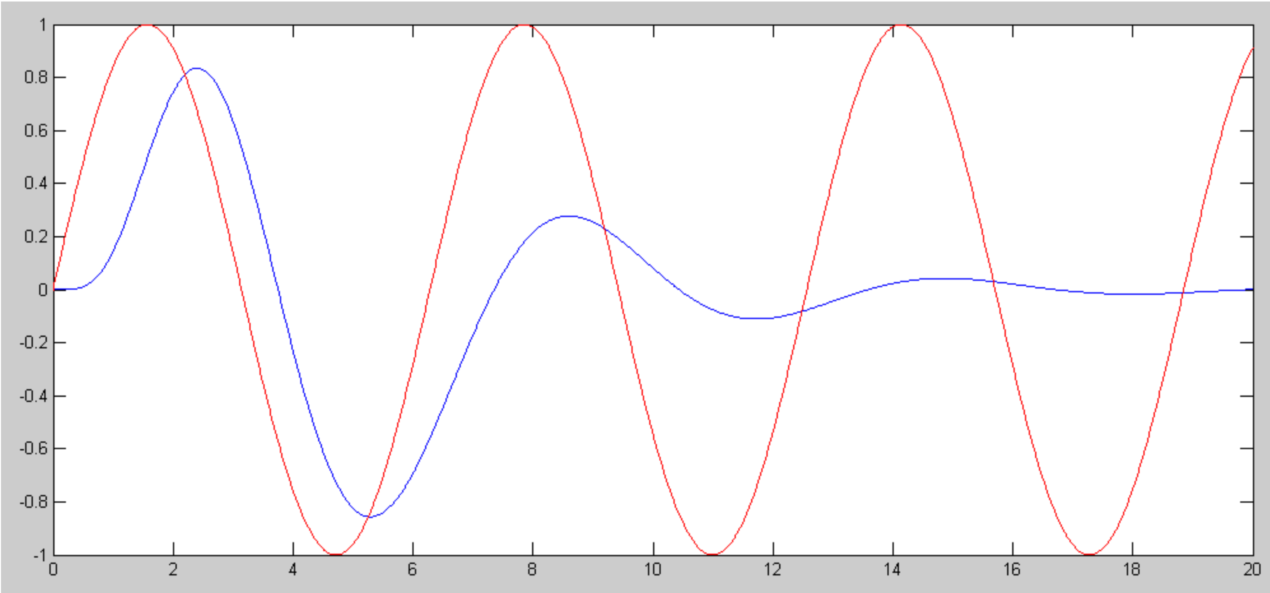
```
R = sin(t);
y = step3(A7-B7u*K7, B7r, C7, 0, t, X0, R);
plot(t,y,t,R,'r')
```



The servo compensator can track a sinusoidal set point at 1 rad/sec

The servo compensator can reject a disturbance at 1 rad/sec

```
R = sin(t);
y = step3(A7-B7u*K7, B7u*100, C7, 0, t, X0, R);
plot(t,y,t,R,'r')
```



It can also

- Track a constant set point,
- While rejecting a sinusoidal disturbance at 1 rad/sec

To do this, define the B matrix to be a 7x2 matrix and the inputs to be a 1x2 matrix

$$\begin{bmatrix} \dot{X} \\ \dot{Z} \end{bmatrix} = \begin{bmatrix} A - BK_x & -BK_z \\ B_z C & A_z \end{bmatrix} \begin{bmatrix} X \\ Z \end{bmatrix} + \begin{bmatrix} 0 & B \\ -1 & 0 \end{bmatrix} \begin{bmatrix} R \\ d \end{bmatrix}$$

```
B7 = [B7r, B7u];
D7 = [0, 0];
R = 0*t + 1;
d = 100*sin(t);

y = step3(A7-B7u*K7, [B7r, B7u], C7, D7, t, X0, [R, d]);

plot(t,y,t,R,'r')
```