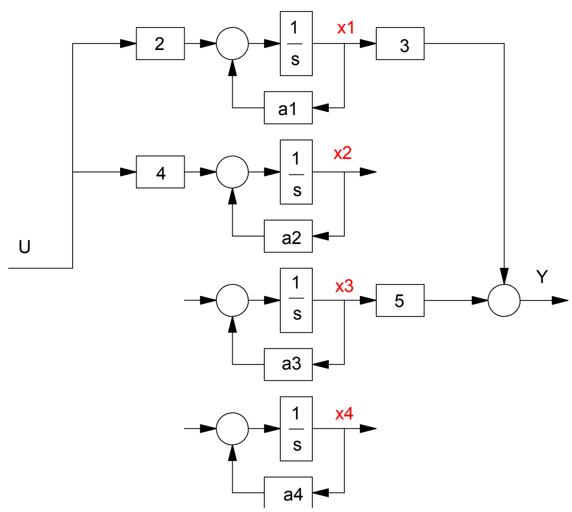
Controllability and Observability

Controllability and Observability are properties of systems which relate to whether the states can be driven to any arbitrary state from a given input (controllable) or whether you can deduce what the system's states are from a given output (observabilty). Controllability and Observability are easiest to see in Jordan form. Consider the following system:

$$sX = \begin{bmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 \\ 0 & 0 & a_3 & 0 \\ 0 & 0 & 0 & a_4 \end{bmatrix} X + \begin{bmatrix} 2 \\ 4 \\ 0 \\ 0 \end{bmatrix} U$$

$$Y = \begin{bmatrix} 3 & 0 & 5 & 0 \end{bmatrix} X$$



Block Diagram of 4th Order System

From the block diagram, it is clear that the input cannot effect states X3 and X4. These states are termed *uncontrollable.* The output Y cannot see states X2 and X4 - meaning no amount of filtering or other mathematical trickery can deduce the value of X2 and X4 based upon measurements of Y. Likewise, states X2 and X4 are *unobservable.*

One way to tell if a system is uncontrollable or unobservable is to count the number of states in the transfer function. The above system is 4th-order. The transfer function from U to Y, however, is 1st-order:

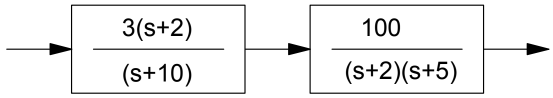
$$Y = \left(\frac{6}{s-a_1} \right) U$$

The missing three states tells you that three states are uncontrollable, unobservable, or both. (Corollary: if the transfer function for an Nth-order system is Nth-order, the system is controllable and observable).

The following are tests to determine if a system is uncontrollable or unobservable.

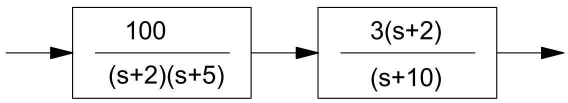
Pole-Zero Cancellation

If a zero which cancels a pole comes before the pole, that state is uncontrollable.



The pole at -2 is uncontrollable due to zero canceling the pole before the pole

If a zero which cancels a pole comes after the pole, that state is unobservable.



The pole at -2 is unobservable due to zero canceling the pole after the pole

Controllability Matrix:

If

$$rank[B, AB, A^2B, A^3B, ...] = N$$

the system is controllable.

Note: The Calley-Hamilton theorem states that A^N is a linear combination of $\{A^0, ..., A^{N-1}\}$, meaning you only have to take the above matrix out to A^{N-1}

The matrix

$$[B, AB, A^2B, A^3B, ..., A^{N-1}B]$$

is termed *the controllability matrix.*

Proof: This is easier to see in discrete-time. Consider the generic system:

$$X(k+1) = AX(k) + BU(k)$$

Assume X(k) = 0 for k<0. At t= 0

$$X_0 = BU_0$$

$$X_1 = AX_0 + BU_1 = ABU_0 + BU_1$$

$$X_2 = AX_1 + BU_2 = A^2BU_0 + ABU_1 + BU_2$$

$$X_N = \left[\begin{array}{ccccc} B & AB & A^2B & \ldots & A^{N-1}B \end{array}\right]\left[\begin{array}{c} U_0 \\ U_1 \\ U_2 \\ \vdots \\ U_{N-1} \end{array}\right]$$

If $\begin{bmatrix} B & AB & A^2B & \dots & A^{N-1}B \end{bmatrix}$ is inevitable (meaning the determinant is non-zero or it is full rank), you can solve for $U_0..U_{N-1}$. You can drive the system to any arbitrary state.

Observability matrix

A system's observabliity matrix is

$$\begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{N-1} \end{bmatrix}$$

If a system's observability matrix is full rank, the system is observable.

Proof: Consider the system with no input:

$$sX = AX$$

$$Y = CX$$

If you take the derivative of the output, you get

$$sY = CsX = CAX$$

$$s^2Y = s(CAX) = CA^2X$$

or

$$\begin{bmatrix} Y \\ sY \\ s^2Y \\ \vdots \\ s^{N-1}Y \end{bmatrix} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{N-1} \end{bmatrix} X$$

If the observability matrix is invertable, you can compute the system's states by measuring the output and computing it's derivatives.

PBH Rank Test

If

$$rank([B, \lambda I - A]) = N$$

for all λ, the system is controllable. Moreover, the values of λ are the modes which are uncontrollable. Note that $(\lambda I - A)$ is full rank for unless λ is an eigenvalue of A. Likewise, you only need to check λ equal to the eigenvalues of the system.

If

$$rank\left(\begin{bmatrix} C \\ \lambda I - A \end{bmatrix}\right) = N$$

for all λ, the system is observable. Moreover, the values of λ which are not observable are the modes which are unobservable.

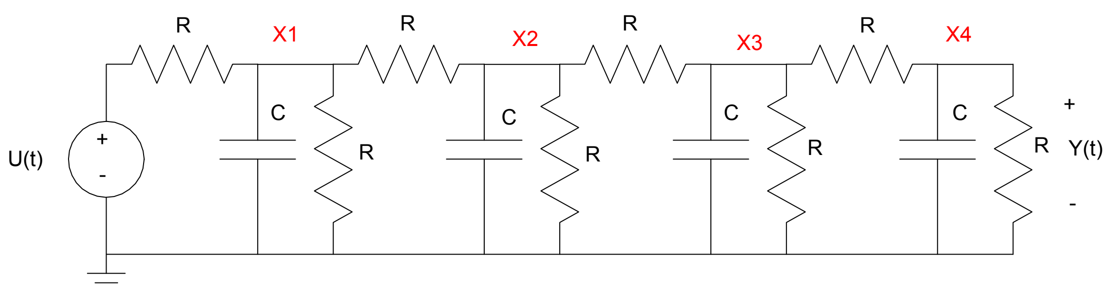
What to do with uncontrollable / unobservable systems?

If a pole is uncontrollable or unobservable and stable, you can live with it and ignore it. You can't do anything about it, but since it's stable, if you leave it alone it will decay on its own.

If a pole is uncontrollable or unobservable and *unstable*, you need a new system. Find another input or output.

Examples:

Is the following system controllable and observable? (heat equation with four states)



$$sX = \begin{bmatrix} -3 & 1 & 0 & 0 \\ 1 & -3 & 1 & 0 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix} X + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} U$$

$$Y = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} X$$

In SciLab (or MATLAB):

```
-->A = [-3,1,0,0;1,-3,1,0;0,1,-3,1;0,0,1,-2]

  - 3.    1.    0.    0.
    1.  - 3.    1.    0.
    0.    1.  - 3.    1.
    0.    0.    1.  - 2.

-->B = [1;0;0;0]

    1.
    0.
    0.
    0.

-->C = [0,0,0,1]
```

```
     0.     0.     0.     1.

-->rank([B, A*B, A*A*B, A*A*A*B])

     4.
```

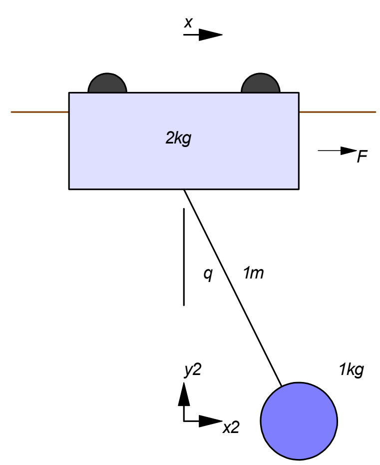
Yes, this system is controllable. From the input, you can drive all four states wherever you like.

```
-->rank([C; C*A; C*A*A; C*A*A*A])

     4.
```

Yes, this system is observable. You can deduce the temperature at all four states just by measuring the output and its derivatives.

Example 2: The dynamics for a cart and pendulum are



:

$$s\begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -19.6 & 0 & 0 \\ 0 & 29.4 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ \theta \\ x' \\ \theta' \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}F + \begin{bmatrix} 0 \\ 0 \\ -2 \\ 3 \end{bmatrix}T$$

Is this system controllable from the torque at the base of the pendulum? (T)

```
-->A = [0,0,1,0;0,0,0,1;0,-19.6,0,0;0,29.4,0,0]

     0.     0.       1.     0.
     0.     0.       0.     1.
     0.   - 19.6     0.     0.
     0.     29.4     0.     0.

-->B = [0;0;-2;3]
```

```
       0.
       0.
     - 2.
       3.

-->rank([B, A*B, A*A*B, A*A*A*B])

       2.
```

No, the system is not controllable: only two modes are controllable. To determine which two, use the PBH test

```
->L = spec(A)

       0
       0
       5.4221767
     - 5.4221767

-->I = eye(4);

-->rank([B, L(1)*I-A])

       3.
```

The pole at s = 0 is not controllable.

```
-->rank([B, L(3)*I-A])

       4.

-->rank([B, L(4)*I-A])

       4.
```

The poles at +/- 5.422 are controllable. This actually makes sense: you can control the angle of the pendulum with the torque (which corresponds to the poles at +/- 5.422). You cannot control the position or velocity of the pendulum by applying a torque to the pendulum.

Moral: Don't try to control the position of a cart by applying a torque. It won't work.

Is this system controllable from the force to the cart (F)?

```
-->B = [0;0;1;-1]

       0.
       0.
       1.
     - 1.

-->rank([B, A*B, A*A*B, A*A*A*B])

       4.
```

Yes, you can balance a cart and pendulum by applying a force to the base. It may not be easy, but it is theoretically possible.

Is this system observable by measuring the beam angle (θ)?

```
-->C = [0,1,0,0]

    0.    1.    0.    0.

-->rank([C; C*A; C*A*A; C*A*A*A])

    2.
```

No - only two of the four states are observable from the angle. The two modes you can see are found from the PBH test:

```
-->rank([C; L(1)*I-A])

    3.
```

The poles at s = 0 are not observable.

```
 -->rank([C; L(3)*I-A])

    4.

-->rank([C; L(4)*I-A])

    4.
```

The poles at +/- 5.422 are observable.

Again, this makes sense. The poles at s=0 correspond to the cart position and velocity. Just looking at the beam angle tells you nothing about these (hence the system is unobservable).

Is the system observable from the cart position?

```
-->C = [1,0,0,0]
    1.    0.    0.    0.

-->rank([C; C*A; C*A*A; C*A*A*A])
    4.
```

This is somewhat surprising, but yes, you can determine all four states just by measuring the cart position and its derivatives. You can even deduce the beam angle just by looking at position.