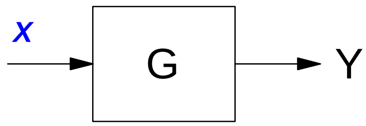
LaPlace Transforms and Dominant Poles

Static and Dynamic Systems:

Signals and Systems deals with trying to describe mathematically the relationship of a system's input (U), it's output (Y), and the system itself (G). This is a very general problem and covers global warming (where the input corresponds to green-house gas emissions, the output is global temperature, and G is the Earth), economics (money supply to inflation rate), and electronics (voltage in to voltage out).



A static system is one where the output and input are identical, save a scaling change. This means that:

- Static systems do not have memory: you don't need to know anything about the previous output or input to determine the output after t=0.
- A sine wave input produces a sine wave output
- A square wave input produces a square wave output
- A random input produces the identical random output, only changed in amplitude.

With a static system you can model the system with a constant, k:

$$Y = k \cdot X$$

A dynamic system, in contrast, requires a differential equation to relate the input and output. This implies:

- Dynamics systems have memory: you have to specify the initial conditions as well as the input to determine the output.
- Dynamic systems change the shape of the input
  - A sine wave input produces a sine wave out, but at a different amplitude and with a delay (phase shift)
  - An input that is not a sine wave is distorted.

With a dynamic system, you need to use a differential equation to model the system:

$$y'' + ay' + by = cx'' + dx' + ex$$

A transfer function is a shorthand way of writing a differential equation. If you assume that all functions are in the form of

$$y = e^{st}$$

then differentiation becomes multiplication by 's'

$$\frac{dy}{dt} = s \cdot e^{st} = sy.$$

With this assumption, the differential equation can be written as a gain, G(s):

$$s^2Y + asY + bY = cs^2X + dsX + eX$$

$$Y = \left(\frac{cs^2+ds+e}{s^2+as+b}\right)X$$

or

$$Y = G(s) \cdot X$$

$$G(s) = \left(\frac{cs^2+ds+e}{s^2+as+b}\right)$$

Steady-State Solution for Sinusoidal Inputs (phasors):

In Circuits II, the problem of finding the output (Y) when you know the input (U) and the system (G) was covered for case when the input was a pure sine wave. In this special case, phasor analysis was used to determine the gain and phase shift. For example, if

$$G(s) = \left(\frac{200}{s^2+20s+100}\right)$$

and

$$x(t) = 4\sin(20t)$$

then y(t) is found by

- Letting s = j20
- Evaluating G(s) at s = j20

$$\left(\frac{200}{s^2+20s+100}\right)_{s=j20} = 0.40\angle -126^0$$

- Finding the output as the gain times the input:

$$Y = (0.4\angle -126^0) \cdot (4\sin(20t))$$

$$\boxed{y(t) = 1.6\sin(20t - 126^0)}$$

Transient Solutions: LaPlace Transforms

If the input is zero for t<0

$$x = x(t) \cdot u(t)$$

where u(t) is the unit step function

$$u(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$$

then LaPlace transforms are used to find y(t). In ECE 343 Signals, you looked at two-sided two-dimensional LaPlace transforms. In this class, life is *much* easier

- t represents time, which is one-dimensional and
- time always goes forward.

Hence, in this class, we look at the mundane case of single-sided one-dimensional LaPlace transforms.

For this case, you really only need a table of four LaPlace transforms to solve any problem. With partial fraction expansion, you can then solve any system.

Table 1: Common LaPlace Transforms		
Name	Time: y(t)	LaPlace: Y(s)
delta (impulse)	$\delta(t)$	1
unit step	u(t)	$\frac{1}{s}$
exponential	$a \cdot e^{-bt}u(t)$	$\frac{a}{s+b}$
damped sinusoid	$2a \cdot e^{-bt}\cos(ct-\theta)u(t)$	$\left(\frac{a\angle\theta}{s+b+jc}\right)+\left(\frac{a\angle-\theta}{s+b-jc}\right)$

Example:

Find the output of a system which satisfies the following differential equation:

y'' + 3'y + 2y = 4x

given that all initial conditions are zero and u is a unit step input,

x(t) = u(t)

Solution:

Convert to LaPlace notation

$$(s^2+3s+2)Y = 4X$$

$$Y = \left(\frac{4}{s^2+3s+2}\right)X$$

Substitute the LaPlace transform for U(s)

$$Y = \left(\frac{4}{s^2+3s+2}\right)\left(\frac{1}{s}\right)$$

Factor and use partial fractions to expand Y(s)

$$Y = \left(\frac{4}{s(s+1)(s+2)}\right)$$

$$Y = \left(\frac{2}{s}\right)+\left(\frac{-4}{s+1}\right)+\left(\frac{2}{s+2}\right)$$

Use the above table to convert back to y(t)

$$\boxed{y(t) = (2-4e^{-t}+2e^{-2t})u(t)}$$

Example 2:

Find the y(t) given that

$$Y(s) = G \cdot X = \left(\frac{15}{s^2+2s+10}\right) \cdot \left(\frac{1}{s}\right)$$

Solution:

Factoring Y(s)

$$Y(s) = \left(\frac{15}{(s)(s+1+j3)(s+1-j3)}\right)$$

Using partial fraction expansion:

$$Y(s) = \left(\frac{1.5}{s}\right) + \left(\frac{0.7906\angle -161.56^0}{s+1+j3}\right) + \left(\frac{0.7906\angle 161.56^0}{s+1-j3}\right)$$

$$y(t) = 1.5 + 1.5812 \cdot e^{-t} \cdot \cos(3t + 161.56^0) \qquad \text{for } t>0$$

Dominant Poles

Poles represent energy and how the energy in the system moves about: if there are N ways to store energy, the system has N poles. In theory, the number of energy states for any system is *very* large - suggesting that you need *very* high-order differential equations to describe any system. Fortunately, just a few poles dominate the response. If you model only includes these dominant poles you'll have

- A fairly accurate model (good),
- That is fairly low-order (also good).

To get the idea of what a dominant pole is, consider the following system:

$$Y = \left(\frac{200}{(s+1)(s+100)}\right)X$$

where

$$x(t) = u(t)$$

To find y(t), replace X with its LaPlace transform

$$Y = \left(\frac{200}{(s+1)(s+100)}\right)\left(\frac{1}{s}\right)$$

Expand using partial fractions

$$Y = \left(\frac{2}{s}\right) + \left(\frac{-2.0202}{s+1}\right) + \left(\frac{0.0202}{s+100}\right)$$

Taking the inverse LaPlace transform

$$y(t) = 2 - 2.0202e^{-t} + 0.0202e^{-100t} \qquad t > 0$$

Here, the pole at -1 is dominantes the pole at -100 for two reasons:

- Its initial condition is 100x larger than the pole at s = -100, and
- Its transient response lasts 100x longer than the pole at s = -100

Hence, the pole at -1 is called 'the dominant pole' and the system can be approximated by

- Keeping the same dominant pole(s), and

- Matching the DC gain

$$Y=\left(\frac{200}{(s+1)(s+100)}\right)X\approx\left(\frac{2}{s+1}\right)X$$

The dominant pole of a system is the pole closest to s=0

- **If the pole is a single-real pole, the system behaves like a 1st-order system**
- **If the pole is a pair of complex conjugate poles, the system behaves like a 2nd-order system**

First-Order approximations

If the system has a single dominant pole, then the system can be approximated as

$$Y=\left(\frac{a}{s+b}\right)X$$

All systems like this behave about the same

- The DC gain is $\frac{a}{b}$
- The transient decays as e^{-bt}

In theory, it takes infinitely long for e^{-bt} to go to zero. Infinity is a difficult number to work with - so instead the 2% settling time is often used:

$$0.02=e^{-bt}$$

Taking the natural log of both sides:

$$-3.912=-bt$$

$$-4\approx-bt$$

$$\boxed{t=\frac{4}{b}}$$ With two degrees of freedom, y

The 2% settling time of a system is 4 / the real part of the dominant pole

Example: Predict what the step response for the following system will look like:

$$Y=\left(\frac{50,000}{(s+3)(s+10)(s+20)(s+50)}\right)X$$

Solution:

- The DC gain of this system is 1.67

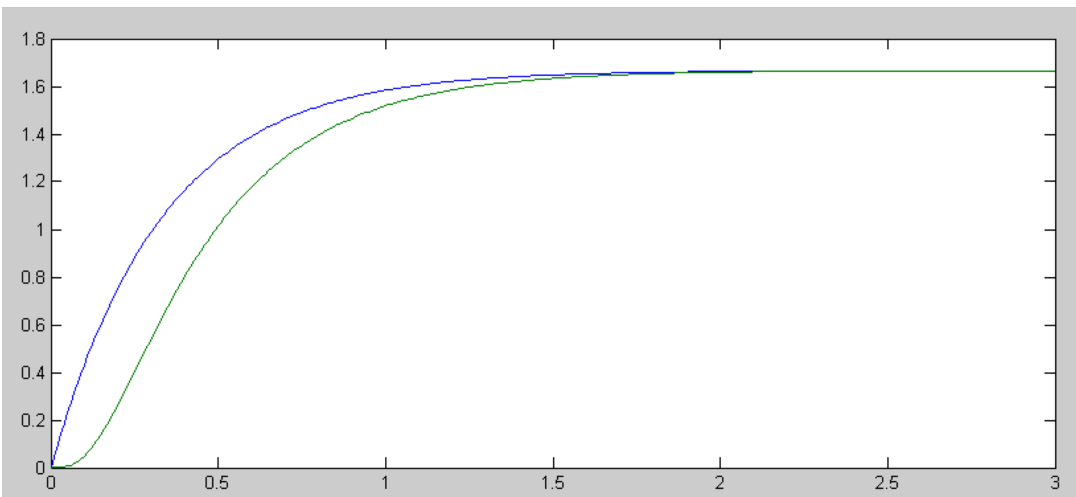
$$\left(\frac{50,000}{(s+3)(s+10)(s+20)(s+50)}\right)_{s=0}=1.67$$

- The dominant pole is s = -3
- The 2% settling time will be 4/3 second, and
- The system behaves similar to

$$G(s) = \left(\frac{50{,}000}{(s+3)(s+10)(s+20)(s+50)}\right) \approx \left(\frac{5}{s+3}\right)$$

Checking with Matlab

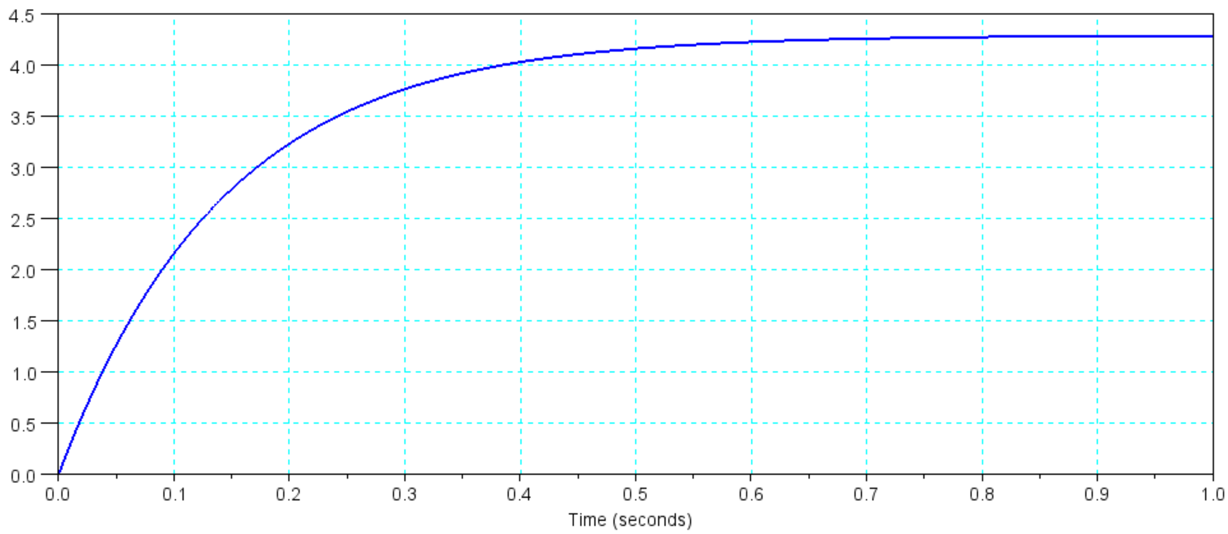
```
>> G4 = zpk([],[-3,-10,-20,-50],50000)

          50000
--------------------------
(s+3) (s+10) (s+20) (s+50)

>> evalfr(G4,0)

    1.6667

>> G1 = zpk([],[-3],5)

  5
-----
(s+3)

>> t = [0:0.01:3]';
>> y1 = step(G1,t);
>> y4 = step(G4,t);
>> plot(t,y1,t,y4);
>>
```



Step response of the 4th-order system (blue) and 1st-order model (green)

As a rule of thumb, poles more than 10x faster than the dominant pole can be ignored. Here, the pole at -10 is borderline - meaning the 1st-order model will be a little off. This is seen in the slight delay in the actual 4th-order system.

Example 2: Find the transfer function for the system with the following response to a unit step input:



Solution: This is a 1st-order system - it behaves like a room warming up or a capacitor charging (no oscillations). So we know it is of the form

$$G(s) = \frac{a}{s+b}$$

DC Gain: The steady-state output is 4.3

$$\left(\frac{a}{s+b}\right)_{s=0} = 4.3$$

The 2% settling time is about 0.57 seconds (ball park)

$$T_{2\%} = \frac{4}{b}$$

$$b = \frac{4}{0.57s} = 7$$

Putting it together:

$$\boxed{G(s) \approx \left(\frac{30.1}{s+7}\right)}$$

2nd-Order Approximations

If the system's dominant poles are a complex conjugate pair, the system behaves like a 2nd-order system

$$G(s) \approx \left(\frac{k \cdot \omega_o^2}{s^2+2\zeta\omega_o s+\omega_o^2}\right) = \left(\frac{k \cdot \left(\sigma^2+\omega_d^2\right)}{(s+\sigma+j\omega_d)(s+\sigma-j\omega_d)}\right)$$

Here you need three parameters to describe the system (pick 3 of the following)

- The DC gain is $G(s)$ as $s \to 0$
- The complex part of the dominant pole is the frequency of oscillation

$$\omega_d = 2\pi f = \frac{2\pi}{T}$$

- The real part of the dominant pole tells you the 2% settling time

$$T_{2\%} = \frac{4}{\sigma}$$

- The angle of the dominant pole tells you the overshoot

$$OS = \frac{a}{b} = \exp\left(\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}\right)$$

where ζ (zeta) is termed *the damping ratio* and is defined as

$$\zeta = \cos(\theta)$$

Example: Determine the step response of

$$Y = \left(\frac{20{,}000}{(s+1+j6)(s+1-j6)(s+50)}\right)X$$

Solution:

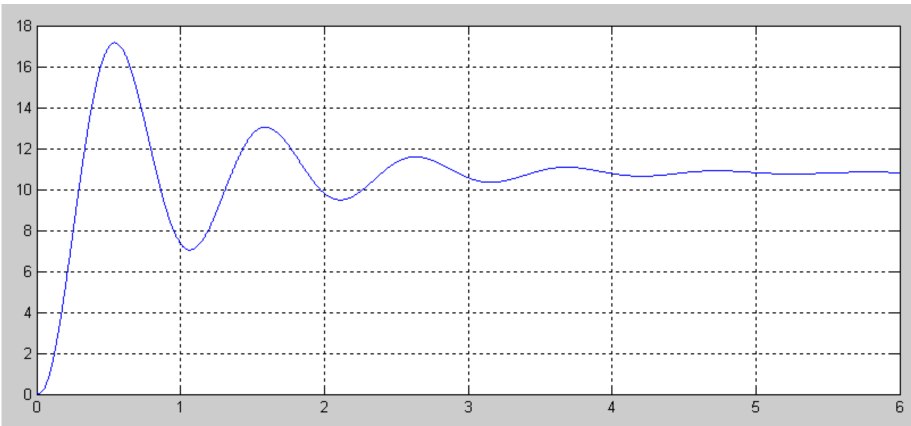
- The dominant poles are $s = -1 \pm j6$
- The DC gain is 10.81

meaning

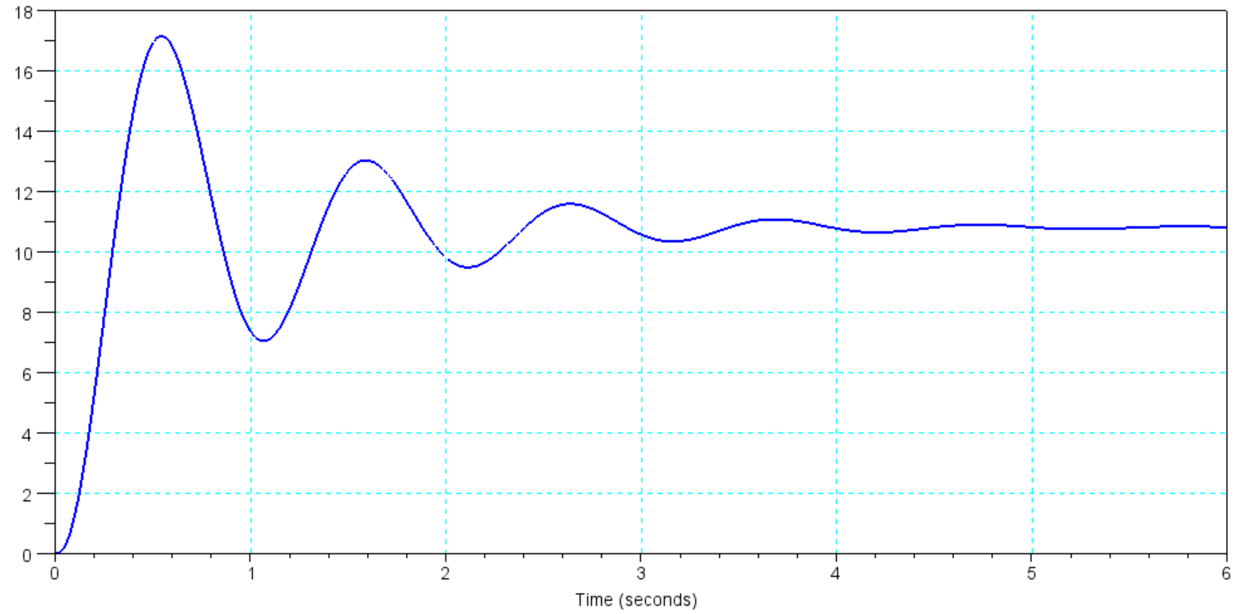
- The system will settle out at 10.81 (the DC gain)
- The 2% settling time will be 4 seconds (4/1)
- The frequency of oscillation will be 6 rad/sec (about 1 Hz)
- The damping ratio is 0.164, meaning
- There will be 59% overshoot for a step input (59% above 10.81)

Checking in MATLAB:

```
>> G3 = zpk([],[-1+j*6,-1-j*6,-50],20000)

        20000
----------------------
(s+50) (s^2 + 2s + 37)

>> t = [0:0.001:6]';
>> y3 = step(G3,t);
>> plot(t,y3);
```



Example 4: Find the transfer function for a system with the following response to a unit step input:



Solution: This is a 2nd-order system (the output rings, meaning you have energy bouncing between two states. For complex poles, you need two pieces of information (real part, complex part, and/or angle)

$$s = \sigma + j\omega_d = \omega_n \angle\theta$$

Real Part of Dominant Pole: The 2% settling time is about 4 seconds

$$\sigma = \frac{4}{t_{2\%}} \approx \frac{4}{4} = 1$$

Complex Part: The frequency of oscillation is

$$\omega_d = \frac{3 \text{ cycles}}{3.2 \text{ seconds}} \cdot 2\pi = 5.89 \frac{\text{rad}}{\text{sec}}$$

Angle: The overshoot is

$$OS = \frac{17.5-10.8}{10.8} = 0.62$$

$$\zeta = 0.1504 = \cos\theta$$

$$\theta = 81.3^0$$

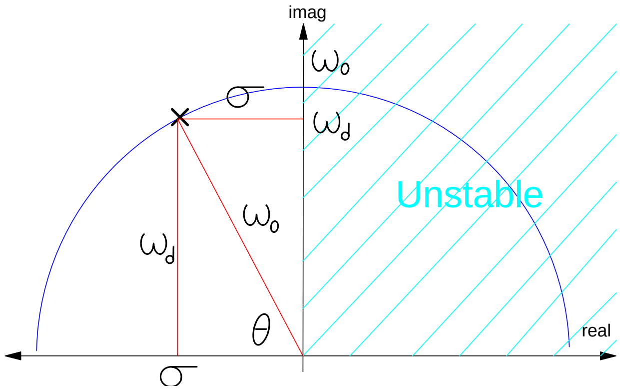
DC gain: The output is 10.8 at steady-state. Since this is a unit step input, the DC gain is 10.8/1 = 10.8.

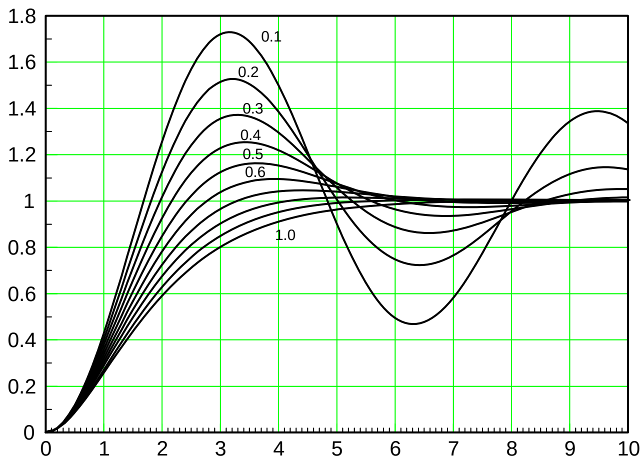
Putting it together

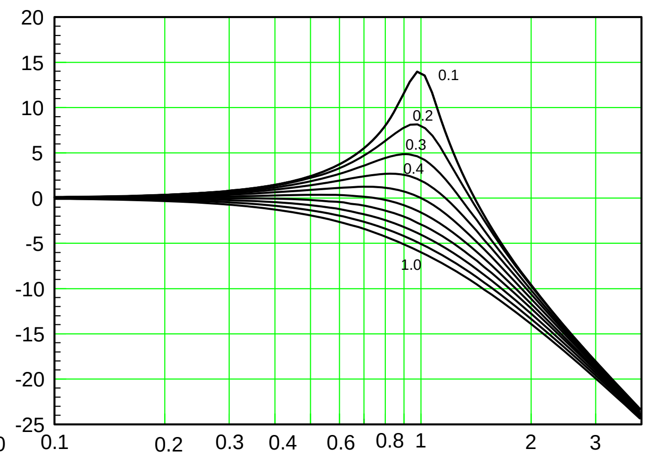
$$\boxed{G(s) \approx \left(\frac{385}{(s+1+j5.89)(s+1-j5.89)}\right)}$$

2nd-Order Approximations:

$$G(s) \approx \left(\frac{k \cdot \omega_o^2}{s^2 + 2\zeta\omega_o s + \omega_o^2}\right) = \left(\frac{k \cdot \left(\sigma^2 + \omega_d^2\right)}{(s + \sigma + j\omega_d)(s + \sigma - j\omega_d)}\right)$$







$\zeta = \cos\theta$	damping ratio	$M_m = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$	max gain
$\%OS = \exp\left(-\left(\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right)\right)$	overshoot	$\frac{1}{2\zeta}$	gain at corner
$\omega_d = \frac{2\pi}{T} = 2\pi f$	damped natural freq		
$T_s = T_{2\%} = \frac{4}{\sigma}$	2% settling time		

zeta	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
%OS	100%	73%	53%	37%	25%	16%	9%	5%	1.5%	0.1%	0%
Mm	inf	5.02	2.55	1.75	1.36	1.15	1.04	1	1	1	1