Solving Difference Equations with z-Trasforms

Home Loan

Assume you borrow $100,000 to buy a house. The loan is for 30 years (360 months), with an interest rate of 6% per year. What is the monthly payment?

This is actually a difference equation - meaning a z-transform problem.

Let

- x(k) be the loan amount at month k.
- p be the monthly payment.

The amount of the loan is then

$$x(k+1) = 1.005 \cdot x(k) - P \cdot u(k-1) + 100,000 \cdot \delta(k)$$

Note that

- The monthly payments are a unit step function: you pay the same each month starting at month #1
- The loan amount is a delta function: you take out the loan only once, at time = k.

Take the z-transform

$$zX = 1.005X - P\left(\frac{1}{z-1}\right) + 100,000$$

Solve for X

$$X = \frac{100{,}000}{z-1.005} - \frac{P}{(z-1)(z-1.005)}$$

Take the inverse z-transform

$$X = \left(\frac{100{,}000}{(z-1.005)} + \frac{200P}{(z-1)} - \frac{200P}{(z-1.005)}\right)$$

Since the numerator is missing a 'z', the output is delayed by one sample

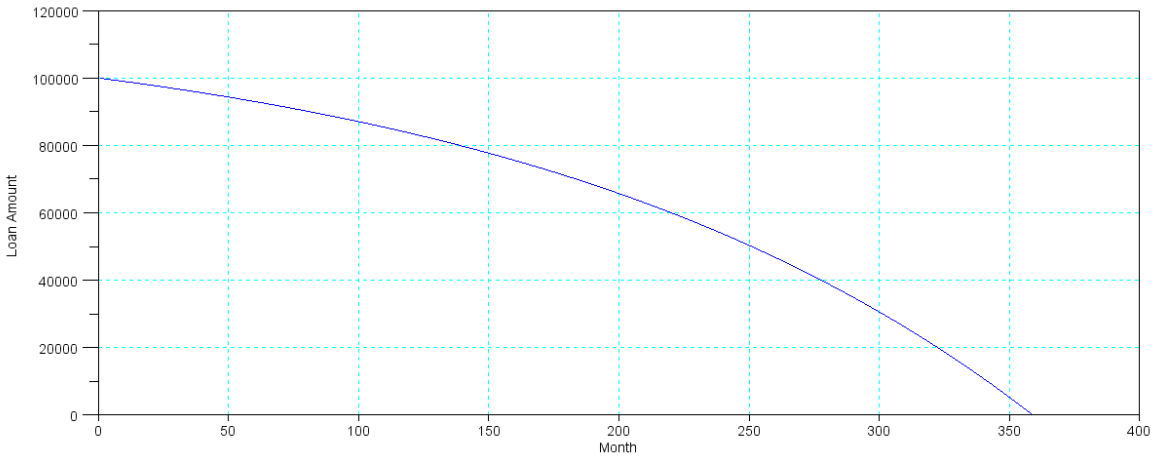
$$x(k) = (100,000 \cdot 1.005^{k-1} + 200P(1 - 1.005^{k-1}))u(k-1)$$

At k = 360 (30 years), the load amount should be zero (the loan is paid off)

$$x(360) = 0 = 599,261 + 100P(1 - 5.9926)$$

$$P = 600.148$$

Your payments are $600.15 per month.



Loan Balance vs. Time for a $100,000 loan paid off over 30 years at 6% Interest

As a sidelight, the total amount you'd pay on this $100,000 loan is

$$P_{total} = 359 \cdot \$600.14$$

$$P_{total} = \$215,453$$

Software Routine

Find the algebraic equation for y(k) for the following C program:

```
// k = 0 (initial values)
x = 3;
y = 4;

for (k=1, k<100, k++) {

   x = 0.9*x + 0.2;

   y = 0.7*y + 0.3*x;

   }
```

Solution: This is a difference equation

$$x_k = 0.9x_{k-1} + 0.2u(k-1) + 3\delta(k)$$

$$y_k = 0.7y_{k-1} + 0.3x_k + 3.1\delta(k)$$

Take the z-transform

$$X = \frac{0.9}{z}X + 0.2\left(\frac{1}{z-1}\right) + 3$$

$$Y = \frac{0.7}{z}Y + 0.3X + 3.1$$

Do some algebra

$$(z - 0.9)X = 0.2\left(\frac{z}{z-1}\right) + 3z$$

$$X = 0.2\left(\frac{z}{(z-1)(z-0.9)}\right) + \left(\frac{3z}{z-0.9}\right)$$

$$(z - 0.7)Y = 0.3zX + 4z$$

$$Y = \left(\frac{0.3z}{z-0.7}\right)\left(\left(\frac{0.2z}{(z-1)(z-0.9)}\right) + \left(\frac{3z}{z-0.9}\right)\right) + \left(\frac{3.1z}{z-0.7}\right)$$

Separate using partial fractions

$$Y = \left(\frac{0.06z^2}{(z-1)(z-0.9)(z-0.7)}\right) + \left(\frac{0.9z^2}{(z-0.9)(z-0.7)}\right) + \left(\frac{3.1z}{z-0.7}\right)$$

$$Y = z\left(\frac{2}{z-1} + \frac{-2.7}{z-0.9} + \frac{0.7}{z-0.7}\right) + z\left(\frac{4.05}{z-0.9} + \frac{-3.15}{z-0.7}\right) + \left(\frac{3.1z}{z-0.7}\right)$$

$$Y = \frac{2z}{z-1} + \frac{1.35z}{z-0.9} + \frac{0.65z}{z-0.7}$$

$$y(k) = \left(2 + 1.35 \cdot (0.9)^k + 0.65 \cdot (0.7)^k\right)u(k)$$

Check in Matlab: The C program (modified for Matlab)

```
x = 3;
y = 4;

Y = zeros(20,1);

for k=1:20
   x = 0.9*x + 0.2;
   y = 0.7*y + 0.3*x;
   Y(k) = y;
   end
```

y(k) as computed

```
k = [1:20]';
Yk = 2 + 1.35*( 0.9 .^ k) + 0.65*( 0.7 .^ k);
```

Comparing the two:

```
[Y,Yk]
```

```
3.67          3.67
3.412         3.412
3.2071        3.2071
3.0418        3.0418
2.906407      2.906407
2.7939172     2.7939172
2.6992311     2.6992311
2.6186019     2.6186019
2.5492475     2.5492475
2.4890768     2.4890768
2.4364969     2.4364969
2.3902767     2.3902767
2.3494497     2.3494497
2.3132451     2.3132451
2.2810389     2.2810389
2.2523179     2.2523179
2.2266541     2.2266541
2.2036862     2.2036862
2.1831059     2.1831059
2.1646471     2.1646471
```

Solving Difference Equations with Real Roots

Find the solution for the following difference equation

$$y_{k+3} - 2.4y_{k+2} + 1.91y_{k+1} - 0.504y_k = 0.7x_{k+1} + 0.2x_k$$

$$x(k) = u(k)$$

Solution: Convert to the z-plane and find the transfer function

$$(z^3 - 2.4z^2 + 1.91z - 0.504)Y = (0.7z + 0.2)X$$

$$Y = \left(\frac{0.7z+0.2}{z^3-2.4z^2+1.91z-0.504}\right)X$$

Plug in the z-transform for x(k)

$$Y = \left(\frac{0.7z+0.2}{z^3-2.4z^2+1.91z-0.504}\right)\left(\frac{z}{z-1}\right)$$

Factor and do a partial fraction expansion

$$Y = z\left(\frac{(0.7z+0.2)}{(z-1)(z-0.9)(z-0.8)(z-0.7)}\right)$$

$$Y = z\left(\frac{150}{z-1} + \frac{-415}{z-0.9} + \frac{380}{z-0.8} + \frac{-115}{z-0.7}\right)$$

Take the invzerse z-transform

$$y(k) = (150 - 415 \cdot 0.9^k + 380 \cdot 0.8^k - 115 \cdot 0.7^k)u(k)$$

Solving Difference Equations with Complex Roots

Find the solution for the following difference equation

$$y_{k+3} - 2.3_{k+2} + 1.85y_{k+1} - 0.511y_k = 0.7x_{k+1} + 0.2x_k$$

$$x(k) = \sin(0.1k)u(k)$$

Solution: Convert to the z-plane and find the transfer function

$$(z^3 - 2.3z^2 + 1.85z - 0.511)Y = (0.7z + 0.2)X$$

$$Y = \left(\frac{0.7z+0.2}{z^3-2.3z^2+1.85z-0.511}\right)X$$

Plug in the z-transform for x(k)

$$X(z) = \left(\frac{z\sin\theta}{z^2-2z\cos\theta+1}\right) = \left(\frac{0.0998z}{z^2-1.99z+1}\right)$$

$$Y = \left(\frac{0.7z+0.2}{z^3-2.3z^2+1.85z-0.511}\right)\left(\frac{0.0998z}{z^2-1.99z+1}\right)$$

Factor and do a partial fraction expansion

$$Y = \left(\frac{0.7z+0.2}{(z-0.7)(z-0.8544\angle 0.3588)(z-0.8544\angle -0.3588)(z-1\angle 0.1)(z-1\angle -0.1)}\right)$$

$$Y = \left(\frac{71.1296}{z-0.7} + \frac{33.4191\angle -0.6348}{z-0.8544\angle 0.3588} + \frac{33.4191\angle 0.6348}{z.-0.8544\angle -0.3588} + \frac{116.3070\angle -2.1379}{z-1\angle 0.1} + \frac{116.3070\angle 2.1379}{z-1\angle -0.1}\right)$$

Take the invzerse z-transform

$$y(k) = (71.1296 \cdot 0.7^k + 66.8382 \cdot 0.8544^k \cdot \cos(0.3599k - 0.6348)$$
$$+232.614 \cdot \cos(0.1k - 2.1379)\ ) \cdot u(k)$$