Circuit Analysis with Forcing Functions

Background

Last lecture covered circuit analysis with LaPlace transforms when there are

- Initial conditions, and
- No input.

Today, let's look at circuit analysis when you have

- Zero initial conditions, and
- A non-zero input.

This is equivalent to having an in put which is zero for t<0

$$v_{in}(t) = f(t)u(t)$$

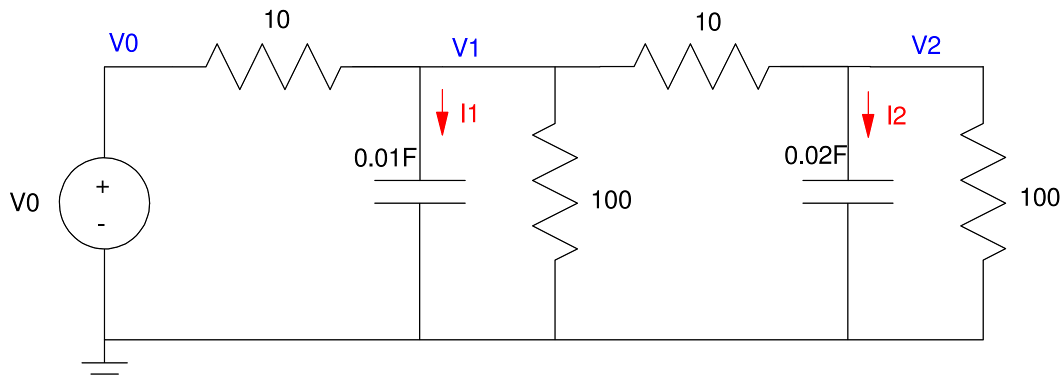
Since the input is zero for t < 0, at t = 0, the states should all be zero.

Again, let's use state-space. It's also easier to explain through examples.

Example 1: 2-Stage RC Filter.

Find y(t) assuming a step input:

$$v_0(t) = u(t)$$



2-Stage RC Filter with an input, V0

The procedure is almost identical to the previous solution, only now with

- Initial conditions are zero, and
- There is an input, V0

Step 1: Define the system states.

This is the voltage across the capacitors and the current through inductors. This defines the energy in the system.

$$X = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

Step 2: Define the change in energy in terms of the input and the system states

$$I_1 = 0.01 \cdot sV_1 = \left(\frac{V_0 - V_1}{10}\right) + \left(\frac{V_2 - V_1}{10}\right) + \left(\frac{0 - V_1}{100}\right)$$

$$I_2 = 0.02 \cdot sV_2 = \left(\frac{0 - V_2}{100}\right) + \left(\frac{V_1 - V_2}{10}\right)$$

Group terms:

$$sV_1 = -21V_1 + 10V_2 + 10V_0$$

$$sV_2 = 5V_1 - 5.5V_2$$

Step 3: Place in matrix (state-space) form

$$s\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} -21 & 10 \\ 5 & -5.5 \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} + \begin{bmatrix} 10 \\ 0 \end{bmatrix} V_0$$

$$Y = V_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} + [0]$$

Solve for the transfer function from V0 to Y in Matlanb:

```
A = [-21,10 ; 5, -5.5]

  -21.0000   10.0000
    5.0000   -5.5000

B = [10 ; 0]

    10
    0

C = [0,1];
D = 0;
G = ss(A,B,C,D);

                  50
G(s) = -------------------
       (s+23.74) (s+2.759)
```

At this point you can solve for Y(s):

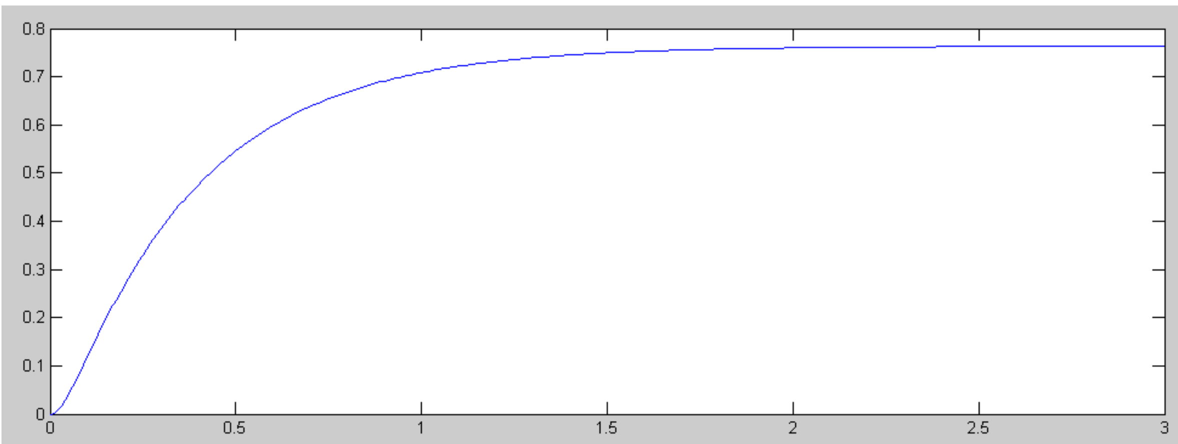
$$Y(s) = G(s) \cdot V_0(s)$$

If the input is a unit step:

$$Y(s) = \left(\frac{10(s+26)}{(s+2.759)(s+23.74)}\right)\left(\frac{1}{s}\right)$$

In Matlab, you can solve using the *step* function

```
>> t = [0:0.01:3]';
>> y = step(G,t);
>> plot(t,y);
```



y(t) for v0(t) = u(t). Note that y(t) = 0 for t<0

You can also find the output for other inputs, but Matlab doesn't have those functions built in (like the step function). If you have a different input, you need to find Y(s) and use the *impulse* command.

Example: Find the response for

$$v_0(t) = \sin(4t)u(t)$$

Solution: Find the LaPlace transfer for $v_0(t)$. From http://tutorial.math.lamar.edu/pdf/Laplace_Table.pdf

$$\sin(at) \leftrightarrow \left(\frac{a}{s^2+a^2}\right)$$

$$\cos(at) \leftrightarrow \left(\frac{s}{s^2+a^2}\right)$$

This gives

$$V_0(s) = \left(\frac{4}{s^2+16}\right)$$

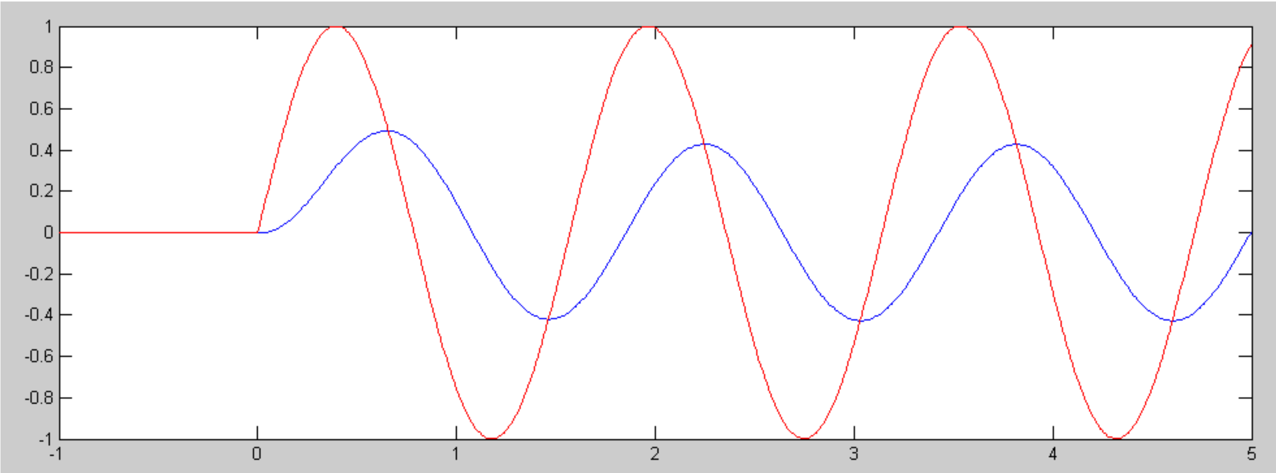
Y(s) is then

$$Y(s) = G(s) \cdot V_0(s)$$

$$Y(s) = \left(\frac{10(s+26)}{(s+2.759)(s+23.74)}\right)\left(\frac{4}{s^2+16}\right)$$

Find Y(s)

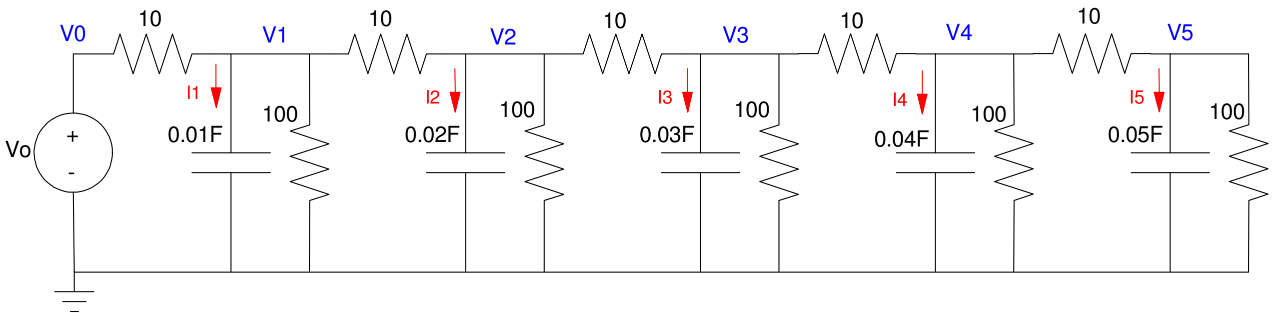
```
Vo = tf(4, [1,0,16])

              4
Vo(s)   =  --------
           s^2 + 16

Y = G * Vo;

zpk(Y)


                            200
Y(s)   =  ------------------------------
          (s+23.74)  (s+2.759)  (s^2 + 16)

t = [0:0.01:5]';
y = impulse(Y,t);

t1 = [-1:0.01:5]';
Vo = sin(4*t1) .* (t1>0);
plot(t,y,'b',t1,Vo,'r')
```



y(t) (blue) and vo(t) (red) for a 4 rad/sec sine wave input.
Note that y(t) = 0 for t<0.

Example 2: 5-stage RC filter.

Find V5(t) assuming $v_0(t) = u(t)$



Again, this follows the previous analysis fairly closely - except that now you have

- No initial conditions, and
- Input Vo.

Step 1: Define the state variables. The energy in the system is defined by

$$X = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \end{bmatrix}$$

Step 2: Define the change in the state variables in terms of the other states

$$I_1 = 0.01sV_1 = \left(\frac{V_0 - V_1}{10}\right) + \left(\frac{0 - V_1}{100}\right) + \left(\frac{V_2 - V_1}{10}\right)$$

$$I_2 = 0.02sV_2 = \left(\frac{V_1 - V_2}{10}\right) + \left(\frac{0 - V_2}{100}\right) + \left(\frac{V_3 - V_2}{10}\right)$$

$$I_3 = 0.03sV_3 = \left(\frac{V_2 - V_3}{10}\right) + \left(\frac{0 - V_3}{100}\right) + \left(\frac{V_4 - V_3}{10}\right)$$

$$I_4 = 0.04sV_4 = \left(\frac{V_3 - V_4}{10}\right) + \left(\frac{0 - V_4}{100}\right) + \left(\frac{V_5 - V_4}{10}\right)$$

$$I_5 = 0.05sV_5 = \left(\frac{V_4 - V_5}{10}\right) + \left(\frac{0 - V_5}{100}\right)$$

Group terms and solve for the derivative

$$sV_1 = -21V_1 + 10V_2 + 10V_0$$

$$sV_2 = 5V_1 - 10.5V_2 + 5V_3$$

$$sV_3 = 3.33V_2 - 7V_3 + 3.33V_4$$

$$sV_4 = 2.5V_3 - 5.25V_4 + 2.5V_5$$

$$sV_5 = 2V_4 - 2.2V_5$$

Place in matrix form

$$s\begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \end{bmatrix} = \begin{bmatrix} -21 & 10 & 0 & 0 & 0 \\ 5 & -10.5 & 5 & 0 & 0 \\ 0 & 3.33 & -7 & 3.33 & 0 \\ 0 & 0 & 2.5 & -5.25 & 2.5 \\ 0 & 0 & 0 & 2 & -2.2 \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \end{bmatrix} + \begin{bmatrix} 10 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} V_0$$

$$Y = V_5 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \end{bmatrix} + [0]$$

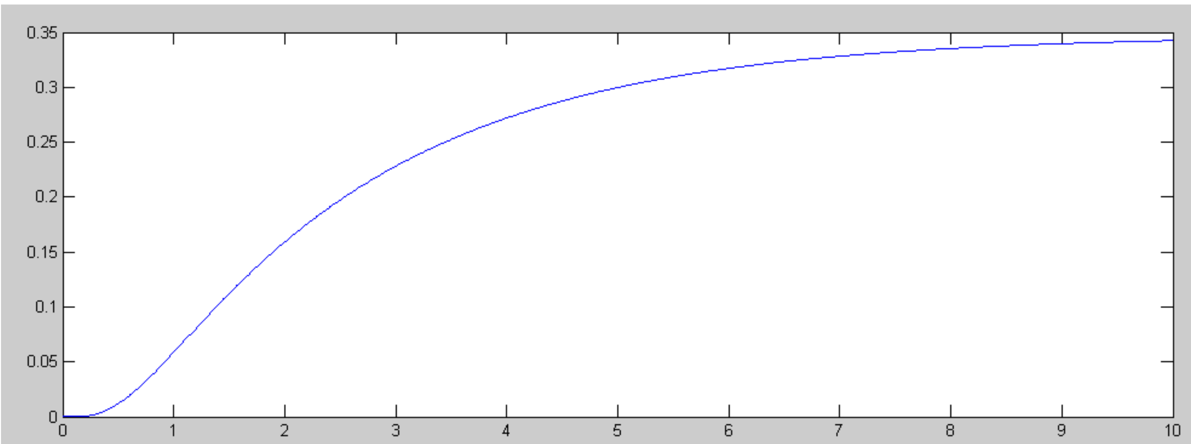
Step 3: Find the transfer function from V0 to Y:

```
A = [-21,10,0,0,0 ; 5,-10.5,5,0,0 ; 0,3.333,-7,3.333,0 ; 0,0,2.5,-5.25,2.5 ; 0,0,0,2,-2.2]

  -21.0000   10.0000         0         0         0
    5.0000  -10.5000    5.0000         0         0
         0    3.3330   -7.0000    3.3330         0
         0         0    2.5000   -5.2500    2.5000
         0         0         0    2.0000   -2.2000

B = [10;0;0;0;0]

    10
     0
     0
     0
     0

C = [0,0,0,0,1];
D = 0;
G = ss(A,B,C,D);
```

```
zpk(G)

                              833.25
G(s) = --------------------------------------------------
       (s+24.76) (s+11.31) (s+6.601) (s+2.822) (s+0.4601)
```

Step 4: Find y(t) for Vo(t) = u(t). This is the *step* command in Matlab:

```
t = [0:0.01:10]';
y = step(G,t);
plot(t,y);
```



Response for a step input: vo(t) = u(t)

Repeat for

$$v_0(t) = \cos(t)u(t)$$

Take the LaPlace transform of vo:

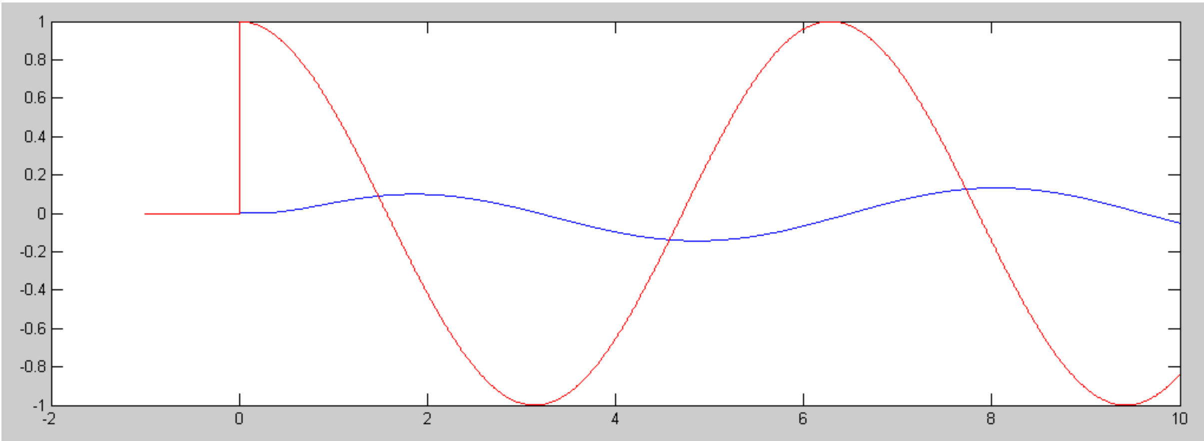
$$V_0(s) = \left(\frac{s}{s^2+1}\right)$$

Y(s) is then

$$Y(s) = G(s) \cdot V_0(s)$$

```
>> Vo = tf([1,0],[1,0,1])

            s
Vo(s) = -------
        s^2 + 1

>> Y = G * Vo;
>> zpk(Y)

                              833.25 s
Y(s) = ------------------------------------------------------------
       (s+24.76) (s+11.31) (s+6.601) (s+2.822) (s+0.4601) (s^2 + 1)
```

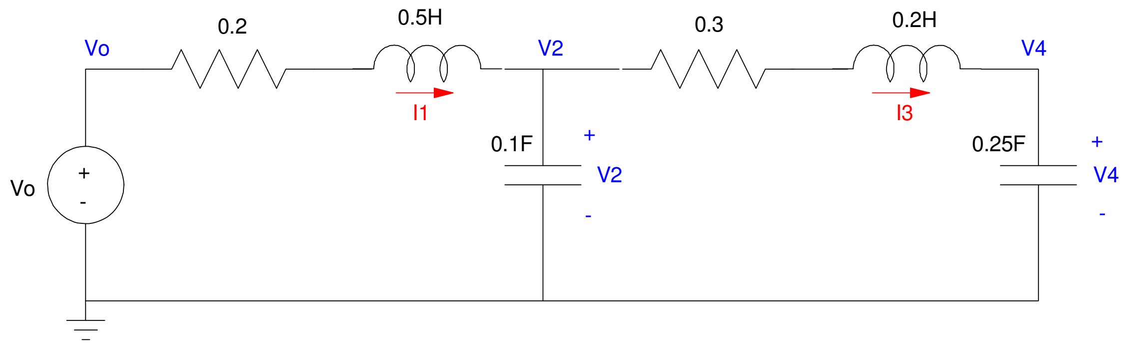
```
y = impulse(Y, t);
t1 = [-1:0.01:10]';
vo = cos(t1) .* (t1>0);
plot(t,y,'b',t1,vo,'r')
```



y(t) and vo(t) for vo(t) = cos(t) u(t)

Example 3: RLC Circuit

Find V4(t) assuming vo(t) = u(t)



Solution: Follow the same procedure as before but with

- No initial conditions, and
- An input, Vo

Step 1: Define the state variables. These define the energy in the system

$$X = \begin{bmatrix} I_1 \\ V_2 \\ I_3 \\ V_4 \end{bmatrix}$$

Step 2: Define the change in the states.

$$v_1 = 0.5sI_1 = (V_0 - 0.2I_1) - V_2$$

$$i_2 = 0.1sV_2 = I_1 - I_3$$

$$v_3 = 0.2sI_3 = V_2 - 0.3I_3 - V_4$$

$$i_4 = 0.25sV_4 = I_3$$

Step 3: Rewrite these equations as

$$sI_1 = -0.4I_1 - 2V_2 + 2V_0$$

$$sV_2 = 10I_1 - 10I_3$$

$$sI_3 = 5V_2 - 1.5I_3 - 5V_4$$

$$sV_4 = 4I_3$$

Place in matrix form

$$s\begin{bmatrix} I_1 \\ V_2 \\ I_3 \\ V_4 \end{bmatrix} = \begin{bmatrix} -0.4 & -2 & 0 & 0 \\ 10 & 0 & -10 & 0 \\ 0 & 5 & -1.5 & -5 \\ 0 & 0 & 4 & 0 \end{bmatrix} \begin{bmatrix} I_1 \\ V_2 \\ I_3 \\ V_4 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} V_0$$

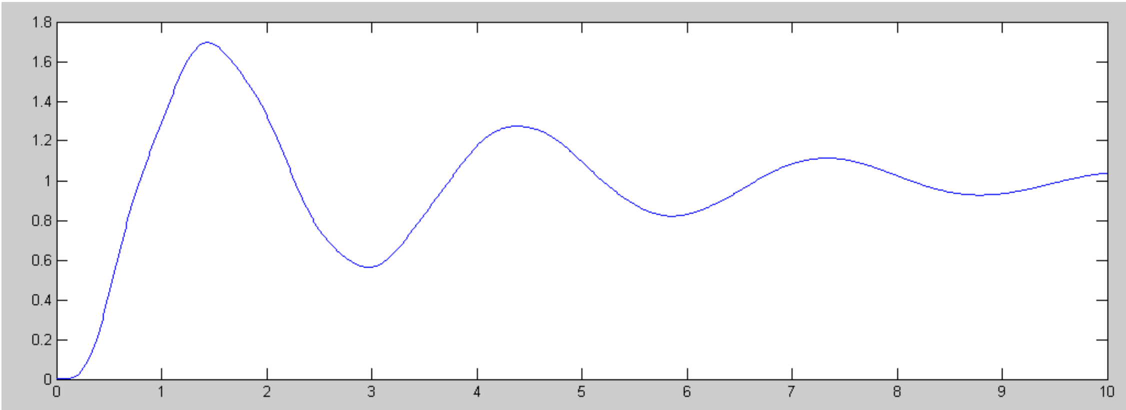
$$Y = V_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} I_1 \\ V_2 \\ I_3 \\ V_4 \end{bmatrix} + [0]$$

Solve in Matlab

```
A = [-0.4,-2,0,0 ; 10,0,-10,0 ; 0,5,-1.5,-5 ; 0,0,4,0]

   -0.4000   -2.0000         0         0
   10.0000         0  -10.0000         0
         0    5.0000   -1.5000   -5.0000
         0         0    4.0000         0

B = [2 ; 0 ; 0 ; 0]

     2
     0
     0
     0

>> C = [0,0,0,1];
>> D = 0;
>> G = ss(A,B,C,D);
>> zpk(G)

                          400
G(s) = --------------------------------------------
       (s^2 + 0.6102s + 4.7) (s^2 + 1.29s + 85.11)
```

Find the step response (vo(t) = u(t))

```
t = [0:0.01:10]';
y = step(G, t);
plot(t,y)
```



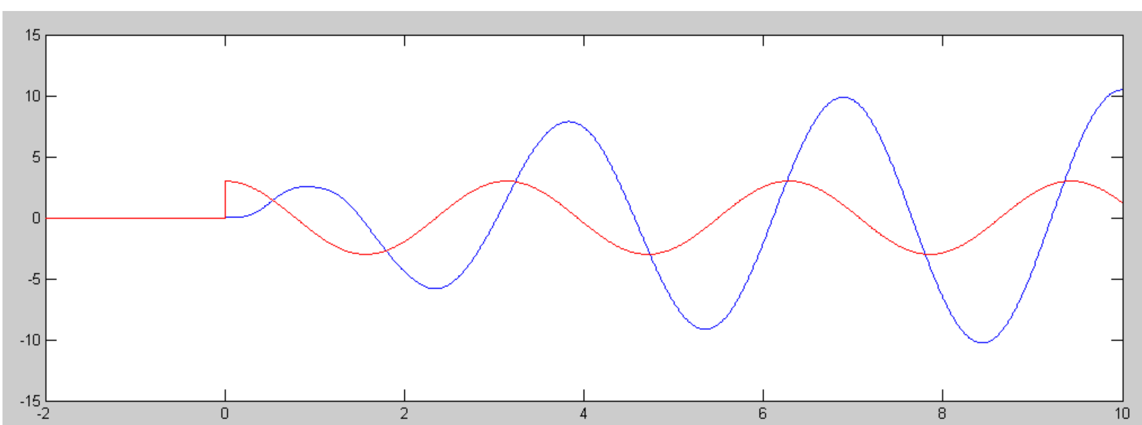
Step Response: vo(t) = u(t)

Repeat for

$$v_0(t) = 3\cos(4t)$$

$$V_0(s) = \left(\frac{3s}{s^2+16}\right)$$

```
>> V0 = tf([3,0],[1,0,4])

        3 s
Vo = -------
      s^2 + 4

>> Y = G * V0;
>> y = impulse(Y,t);
>> t1 = [-2:0.01:10]';
>> vo = 3* cos(2*t1) .* (t1>0);
>> plot(t,y,'b',t1,vo,'r')
```



y(t) (blue) and vo(t) (red) for vo(t) = 3 cos(2t) u(t)