Natural Response of Differential Equations

Background

LaPlace transforms allow you to solve differential equations with non-periodic inputs. The simplest non-periodic input is

$$x(t) = 0$$

In this case, the response depends upon the initial coniditions, and is termed the *Natural Response*.

To to this, use the following table of LaPlace transforms

$$x(t) \leftrightarrow X(s)$$

$$\frac{dx}{dt} = x' \leftrightarrow sX - x(0)$$

$$\frac{d^2x}{dt^2} = x'' \leftrightarrow s^2X - sx(0) - x'(0)$$

$$\frac{d^3x}{dt^3} = x''' \leftrightarrow s^3X - s^2x(0) - sx'(0) - x''(0)$$

etc.

along with

$$\delta(t) \leftrightarrow 1$$

$$u(t) \leftrightarrow \left(\frac{1}{s}\right)$$

$$e^{-at}u(t) \leftrightarrow \left(\frac{1}{s+a}\right)$$

$$2e^{-at}\cos(bt - \theta)u(t) \leftrightarrow \left(\frac{1\angle\theta}{s+a+jb}\right) + \left(\frac{1\angle-\theta}{s+a-jb}\right)$$

Natural Response for Systems with Real Roots

Example 1: Find y(t) given that (using prime notation for derivatives)

$$y' + 3y = 0$$

$$y(0) = 2$$

Solution: Take the LaPlace tranform

$$(sY - y(0)) + 3Y = 0$$

$$sY - 2 + 3Y = 0$$

$$(s + 3)Y = 2$$

$$Y = \left(\frac{2}{s+3} \right)$$

Now take the inverse LaPlace transform

$$y(t) = 2e^{-3t}u(t)$$

As a check, you can plot this in Matlab. There are two ways to input a system into Matlab

```
G = zpk(zeros, poles, gain)
G = tf(num, den)
```

For this system

```
>> G = zpk([], -3, 2)

  2
-----
(s+3)

>> G = tf(2, [1, 3])


  2
-----
s + 3
```

The impulse response finds y(t) (the LaPlace transform for an impulse is 1.000 so it doesn't change the LaPlace transform). Computing the two resposnes and comparing them:

```
G = zpk([], -3, 2)

  2
-----
(s+3)

t = [0:0.01:2]';

y = impulse(G, t);

y1 = 2*exp(-3*t);

N = [1:10:length(t)]';

plot(t,y,'-',t(N),y1(N),'r+')
```
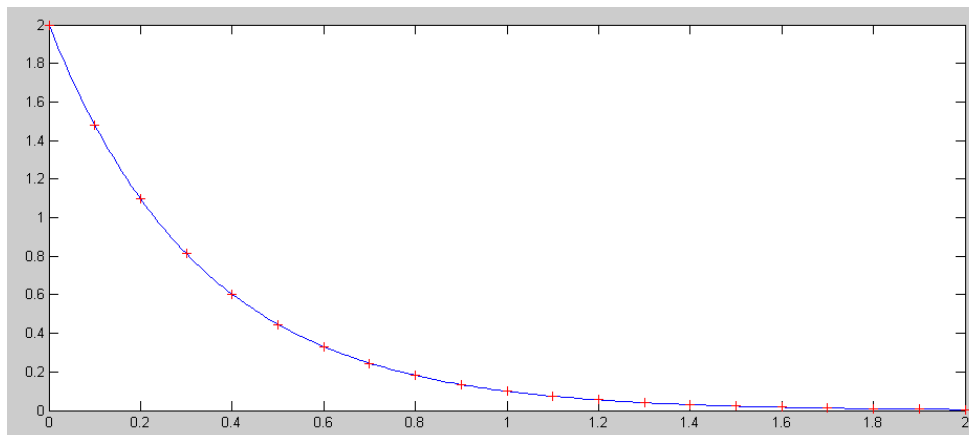
The two match.

Natural Response for y(t): Matlab's Solution (blue) and Hand Solution (red)

Example 2: Higher-order differential equation. Find y(t) given

$$y''' + 6y'' + 11y' + 6y = 0$$

$$y(0) = 5$$

$$y'(0) = 2$$

$$y''(0) = 1$$

Take the LaPlace transform

$$(s^3Y - s^2y(0) - sy'(0) - y''(0)) + 6(s^2Y - sy(0) - y'(0)) + 11(sY - y(0)) = 6Y = 0$$

Group terms

$$(s^3 + 6s^2 + 11s + 6)Y = (5s^2 + 2s + 1) + 6(5s + 2) + 11(5)$$

$$(s^3 + 6s^2 + 11s + 6)Y = (5s^2 + 32s + 68)$$

$$Y = \left(\frac{5s^2 + 32s + 68}{s^3 + 6s^2 + 11s + 6} \right)$$

Take the inverse-LaPlace transform.

Unfortunately, this isn't in our table of LaPlace transforms. So, factor the denominator. In Matlab, this is the command *roots* and *poly*

```
>> roots([1,6,11,6])

    -3.0000
    -2.0000
    -1.0000
```

```
>> poly([-1,-2,-3])

    1       6      11       6
```

giving

$$\left(\frac{5s^2+32s+68}{s^3+6s^2+11s+6}\right) = \left(\frac{5s^2+32s+68}{(s+1)(s+2)(s+3)}\right)$$

Take the partial fraction expansion:

$$\left(\frac{5s^2+32s+68}{(s+1)(s+2)(s+3)}\right) = \left(\frac{a}{s+1}\right) + \left(\frac{b}{s+2}\right) + \left(\frac{c}{s+3}\right)$$

Since these are identical, they all blow up to infinity at the same rate. Evaluating at the singularities:

$$\left(\frac{5s^2+32s+68}{(s+1)(s+2)(s+3)}\right)_{s\to-1} = \left(\left(\frac{a}{s+1}\right) + \left(\frac{b}{s+2}\right) + \left(\frac{c}{s+3}\right)\right)_{s\to-1}$$

On the right side, the first term blows up, making the other terms irrelevant

$$\left(\frac{5s^2+32s+68}{(s+1)(s+2)(s+3)}\right)_{s\to-1} = \left(\frac{a}{s+1}\right)_{s\to-1}$$

Cancelling the (s+1) terms

$$\left(\frac{5s^2+32s+68}{(s+2)(s+3)}\right)_{s\to-1} = (a)_{s\to-1}$$

$$a = 20.5$$

Repeat for b

$$b = \left(\frac{5s^2+32s+68}{(s+1)(s+3)}\right)_{s\to-2}$$

$$b = -24$$

and c

$$c = \left(\frac{5s^2+32s+68}{(s+1)(s+2)}\right)_{s\to-3}$$

$$c = 8.5$$

results in

$$Y = \left(\frac{20.5}{s+1}\right) + \left(\frac{-24}{s+2}\right) + \left(\frac{8.5}{s+3}\right)$$

Take the inverse LaPlace transform

$$y(t) = (20.5e^{-t} - 24e^{-2t} + 8.5e^{-3t})u(t)$$
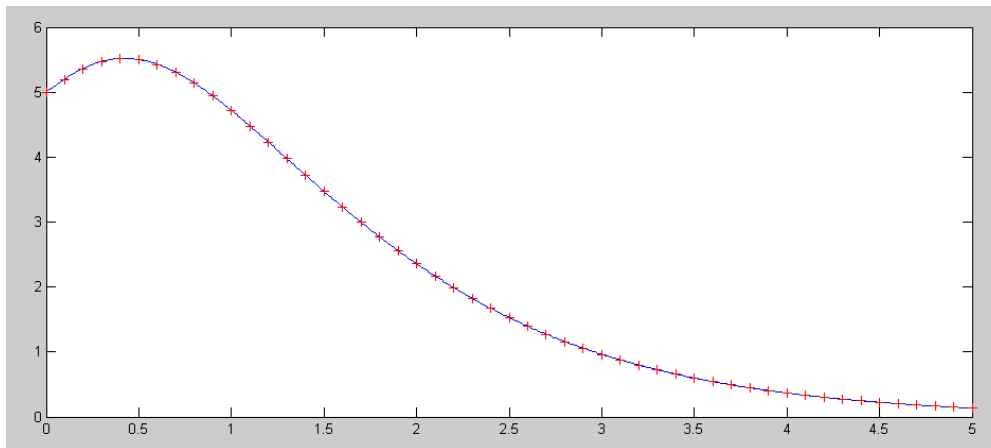
Checking in Matlab:

```
t = [0:0.01:5]';
G = tf([5,32,68],[1,6,11,6]);

y = impulse(G, t);

y1 = 20.5*exp(-t) - 24*exp(-2*t) + 8.5*exp(-3*t);

N = [1:10:length(t)]';
plot(t,y,'-',t(N),y1(N),'r+')
```


Natural Response for y(t): Matlab's Solution (blue) and Hand Solution (red)

Solution with Repeated Roots:

If you have repeated roots, our table doesn't work. So, change the problem. For example, find the solution to the following differential equation:

$$y'' + 2y' + y = 0$$

with the initial conditions

$$y(0) = 5$$

$$y'(0) = 1$$

Solution: Take the LaPlace transform

$$(s^2 Y - sy(0) - y'(0)) + 2(sY - y(0)) + Y = 0$$

$$(s^2 Y - s - 5) + 2(sY - 5) + Y = 0$$

Solve for Y

$$(s^2 + 2s + 1)Y = (s + 15)$$

$$Y = \left(\frac{s+15}{s^2+2s+1} \right)$$

Factor:

$$Y = \left(\frac{s+15}{(s+1)^2} \right)$$

This isn't in our table.

One solution is to use a larger table. Another solution is to change the problem so that

- It is in a form where we know the solution, and
- The flavor of the problem doesn't change significantly.

Change the problem to

$$Y \approx \left(\frac{s+15}{(s+0.99)(s+1.01)} \right)$$

Now use partial fractions

$$Y = \left(\frac{a}{s+0.99} \right) + \left(\frac{b}{s+1.01} \right)$$

$$a = \left(\frac{s+15}{(s+1.01)} \right)_{s \to -0.99} = 700.5$$

$$b = \left(\frac{s+15}{(s+0.99)} \right)_{s \to -1.01} = -699.5$$

resulting in

$$Y \approx \left(\frac{700.5}{s+0.99} \right) + \left(\frac{-699.5}{s+1.01} \right)$$

and

$$y(t) \approx (700.5e^{-0.99t} - 699.5e^{-1.01t})u(t)$$
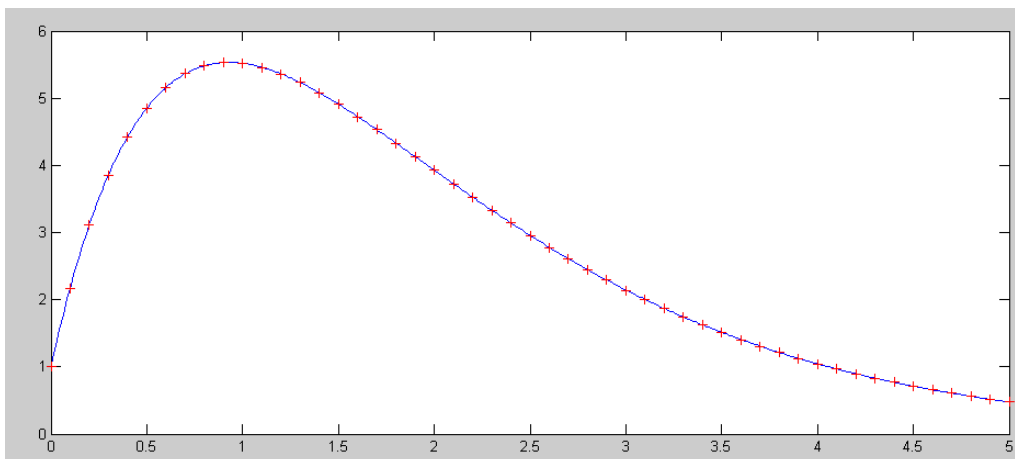
Checking in Matlab:

```
G = zpk(-15, [-1, -1], 1)

(s+15)
-------
(s+1)^2

t = [0:0.01:5]';
y = impulse(G, t);

y1 = 700.5*exp(-0.99*t) - 699.5*exp(-1.01*t);

N = [1:10:length(t)]';
plot(t,y,'-',t(N),y1(N),'r+')
```

Natural Response for y(t): Matlab's Solution (blue) and Hand Solution (red)

Comment: What we did is a common engineering solution: change the problem slightly so that it's easier to solve. In this case, we changed the differential equation from

$$y'' + 2y' + 1 = 0$$

to

$$y'' + 2y' + 0.9999 = 0$$

From a practical standpoint, the differential equation describes a dynamic system where the coefficients are functions of mass, resistances, capacitances, etc. Likewise, the numbers won't be exact (i.e. there are always modeling errors.) Who's to say which differential equation is more accurate given that there are modeling errors?

Likewise, I would argue that repeated roots never happen (the probability of a point is zero). If you encounter repeated roots, change the problem slightly.

Natural Response with Complex Roots

If you have complex roots, nothing changes as long as you don't mind using complex numbers when doing partial fraction expansion. You just use the last entry in our LaPlace transform table:

$$2e^{-at}\cos{(bt - \theta)}u(t) \leftrightarrow \left(\frac{1\angle\theta}{s+a+jb}\right) + \left(\frac{1\angle-\theta}{s+a-jb}\right)$$

Example: Find the solution to the following differential equaiton

$$y''' + 6y'' + 18y' + 40y = 0$$

$$y(0) = 10$$

$$y''(0) = y'(0) = 0$$

Solution: Take the LaPlace transform

$$(s^3Y - s^2y(0)) + 6(s^2Y - sy(0)) + 18(sY - y(0)) + 40Y = 0$$

$$(s^3 + 6s^2 + 18s + 40)Y - (s^2 + 6s + 18) = 0$$

Solve for Y

$$Y = \left(\frac{s^2+6s+18}{s^3+6s^2+18s+40} \right)$$

Factor:

$$Y = \left(\frac{s^2+6s+18}{(s+1+j3)(s+1-j3)(s+4)} \right)$$

Use partial fraction expansion

$$Y = \left(\frac{a}{s+1+j3} \right) + \left(\frac{b}{s+1-j3} \right) + \left(\frac{c}{s+4} \right)$$

$$a = \left(\frac{s^2+6s+18}{(s+1-j3)(s+4)} \right)_{s \to -1-j3} = 0.4969\angle 63.4^0$$

$$b = \left(\frac{s^2+6s+18}{(s+1+j3)(s+4)} \right)_{s \to -1+j3} = 0.4969\angle -63.4^0$$

$$c = \left(\frac{s^2+6s+18}{(s+1+j3)(s+1-j3)} \right)_{s \to -4} = 0.5555$$

meaning

$$Y = \left(\frac{0.4969\angle -63.4^0}{s+1+j3} \right) + \left(\frac{0.4969\angle 63.4^0}{s+1-j3} \right) + \left(\frac{0.5555}{s+4} \right)$$

Taking the inverse LaPlace transform:

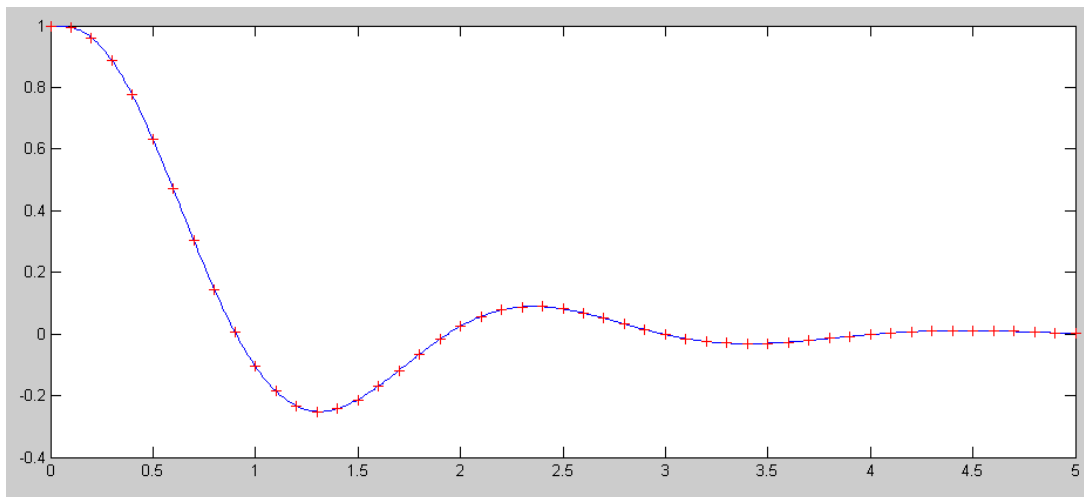$$y(t) = (0.9938e^{-t}\cos(3t - 63.4^0) + 0.5555e^{-4t})u(t)$$

Checking in Matlab:

```
G = tf([1,6,18],[1,6,18,40])

     s^2 + 6 s + 18
----------------------
s^3 + 6 s^2 + 18 s + 40

y = impulse(G, t);

y1 = 0.9938*exp(-t) .* cos(3*t - 1.107) + 0.555*exp(-4*t);
N = [1:10:length(t)]';
plot(t,y,'-',t(N),y1(N),'r+')
```

Natural Response for y(t): Matlab's Solution (blue) and Hand Solution (red)