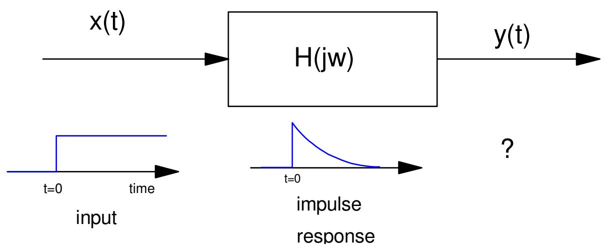
LaPlace Transform

Discussion

Right now, we're looking at the problem of finding the output of a filter with an input which is *not* periodic.



Problem: Find the output of a filter (differential equation) which has an input which is not periodic.

One way to solve this problem is convolution:

$$y(t) = h(t) ** x(t)$$

This is a really hard way to solve the problem. There has to be a better way.

Recall from phasors, if x(t) is a pure sine wave, the problem is easy. Using phasor notation:

$$Y = H \cdot X$$

If x(t) is periodic, then Fourier transforms let you convert this to a bunch of problems, each of which is a phasor problem. In this case

$$Y(j\omega) = H(j\omega) \cdot X(j\omega)$$

where ω is at harmonics of the fundamental frequency

$$\omega = n\omega_0 = \left(\frac{2\pi n}{T}\right)$$

If you work in the frequency domain,

Output = Gain t* Input

LaPlace transforms are similar to Fourier transforms - only they work for non-periodic inputs. Recall that the Fourier transform for a function, x(t), is

$$X_n = \frac{1}{T}\int_T x(t) \cdot e^{-jn\omega_0 t} \cdot dt$$

and its inverse Fourier transform was

$$x(t) = \sum_n X_n e^{jn\omega_0 t}$$

With LaPlace transforms we're dealing with inputs which are *not* periodic and are zero for $t < 0$ (i.e. the system is causal: the output can't happen before the input.)

The basic assumption behind Fourier transforms is that all functions are in the form of

$$y(t) = ae^{j\omega t} = ae^{jn\omega_0 t}$$

This results in differentiation becoming multiplication by jw. With LaPlace transforms, we assume all funcitons are of the form

$$y(t) = ae^{(\sigma+j\omega)}t = ae^{st}$$

With this assumption, Fourier transforms

$$X_n = \frac{1}{T}\int_T x(t) \cdot e^{-jn\omega_0 t} \cdot dt$$

become LaPlace transforms:

$$X(s) = \int_{-\infty}^{\infty} x(t) \cdot e^{-st} \cdot dt$$

The inverse Fourier trasnsform

$$x(t) = \sum_n X_n e^{jn\omega_0 t}$$

becomes the inverst LaPlace transform:

$$x(t) = L^{-1}(X(s)) = \frac{1}{2\pi j}\int_{-j\infty}^{+j\infty} X(s) \cdot e^{st} \cdot \cdot ds$$

Once you convert a system to LaPlace transforms, the output is the gain times the input

$$Y(s) = H(s) \cdot X(s)$$

just like it is with Fourier transforms. But first, you need to find the LaPlace transform for the input, x(t).

LaPlace Transforms for Different Functions:

Delata Function:

$$x(t) = \delta(t)$$

$$X(s) = \int_{-\infty}^{\infty} e^{-st} \cdot \delta(t) \cdot dt$$

$$X(s) = e^{0} = 1$$

The LaPlace transform for a delta function is one

$$\delta(t) \leftrightarrow 1$$

Step Function

$$x(t) = u(t)$$

$$X(s) = \int_{-\infty}^{\infty} e^{-st} \cdot (u(t)) \cdot dt$$

$$X(s) = \int_{0}^{\infty} e^{-st} \cdot dt$$

$$X(s) = \left(\tfrac{-1}{s} \cdot e^{-st}\right)_{0}^{\infty}$$

Assuming the function goes to zero as time goes to infinity

$$X(s) = \tfrac{1}{s}$$

The LaPlace transform for a step function is 1/s:

$$u(t) \leftrightarrow \left(\tfrac{1}{s}\right)$$

Decaying Exponential

$$x(t) = e^{at}u(t)$$

$$X(s) = \int_{-\infty}^{\infty} e^{-st} \cdot (e^{-at}u(t)) \cdot dt$$

$$X(s) = \int_{0}^{\infty} e^{-(s+a)t} \cdot dt$$

$$X(s) = \left(\tfrac{-1}{s+a} \cdot e^{-(s+a)t}\right)_{t=0}^{t=\infty}$$

Assuming the function goes to zero as time goes to infinity

$$X(s) = \left(\tfrac{1}{s+a}\right)$$

The LaPlace transform for a decaying exponential is $\left(\frac{1}{s+a}\right)$

$$e^{-at}u(t) \leftrightarrow \left(\frac{1}{s+a}\right)$$

Damped Sinusoid:

$$x(t) = ae^{-\sigma t}\cos(\omega t + \theta)u(t)$$

This is actually a special case of the previous solution. Asume 'a' is complex:

$$a = \sigma + j\omega$$

along with it's complex conjugate

$$a^* = \sigma - j\omega$$

Then, the LaPlace transform is:

$$X(s) = \left(\frac{r\angle\theta}{s+\sigma+j\omega}\right) + \left(\frac{r\angle-\theta}{s+\sigma-j\omega}\right)$$

The original time function was

$$x(t) = (r\cdot e^{j\theta})(e^{-(\sigma+j\omega t)}) + (r\cdot e^{-j\theta})(e^{-(\sigma-j\omega t)})$$

$$= re^{-\sigma t}\cdot(e^{j(\omega t-\theta)} + e^{-j(\omega t-\theta)})$$

$$= 2re^{-\sigma t}\cdot\left(\frac{e^{j(\omega t-\theta)}+e^{-j(\omega t-\theta)}}{2}\right)$$

$$= 2r\cdot e^{-\sigma t}\cdot\cos(\omega t - \theta)u(t)$$

The LaPlace transform of a damped sinusoid is

$$2r\cdot e^{-\sigma t}\cdot\cos(\omega t-\theta)u(t) \leftrightarrow \left(\frac{r\angle\theta}{s+\sigma+j\omega}\right) + \left(\frac{r\angle-\theta}{s+\sigma-j\omega}\right)$$

Other functions have LaPlace transforms - but these are the main ones you need:

Table of LaPlace Transforms

time domain <-> frequency domain

$$\delta(t) \leftrightarrow 1$$

$$u(t) \leftrightarrow \left(\frac{1}{s}\right)$$

$$e^{-at}u(t) \leftrightarrow \left(\frac{1}{s+a}\right)$$

$$2r \cdot e^{-\sigma t} \cdot \cos(\omega t - \theta)u(t) \leftrightarrow \left(\frac{r\angle\theta}{s+\sigma+j\omega}\right) + \left(\frac{r\angle-\theta}{s+\sigma-j\omega}\right)$$

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