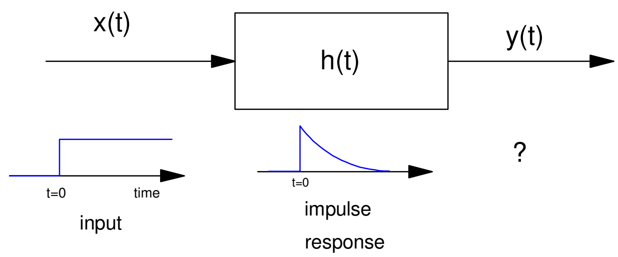
Convolution

So far, we know how to solve differential equations which have

- Sinusoidal inputs,
- A sum of sinusoidal inputs, and
- Periodic inputs.

Now let's look at how to solve differential equations when the input isn't periodic.



Background

Given a function, y(t), its value at t=0 can be expressed in terms of a delta funciton as

$$\int_{-\infty}^{+\infty} y(t) \cdot \delta(t) \cdot dt = y(0)$$

Its value for all time can be evaluated as

$$y(t) = \int_{-\infty}^{\infty} y(\tau) \cdot \delta(t-\tau) \cdot d\tau$$

Suppose h(t) = 1, meaning you have a wire shorting the input to the output so that

$$y(t) = x(t)$$

y(t) can be found using

$$y(t) = \int_{-\infty}^{\infty} x(\tau) \cdot \delta(t-\tau) \cdot d\tau = x(t)$$

Now suppose y(t) and x(t) are related by a differential equation whose inpulse response is h(t). Now, y(t) will come from

$$y(t) = \int_{-\infty}^{\infty} x(\tau) \cdot h(t-\tau) \cdot d\tau$$

This is called *convolution.* The symbol for convolotion is two stars:

$$y(t) = x(t) * *h(t)$$

Graphical Interpritation of Convolution

Let's start with the differential equation we used before:

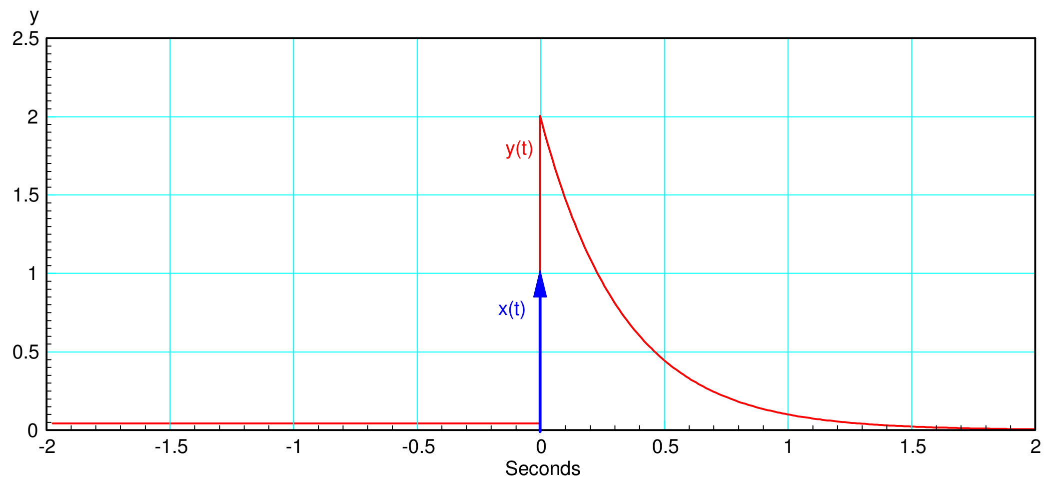
$$\frac{dy}{dt} + 3y = 2x$$

which has the impulse response

$$y(t) = 2e^{-3t} \cdot u(t)$$

Call this h(t) (the impulse response)

$$h(t) = 2e^{-3t} \cdot u(t)$$



Impulse Response: h(t)

The generalized response to an in put x(t) is.

$$y(t) = \int_{-\infty}^{\infty} x(\tau) \cdot h(t-\tau) \cdot d\tau$$

or

$$y(t) = \int_{-\infty}^{\infty} x(t-\tau) \cdot h(\tau) \cdot d\tau$$

(either way works)

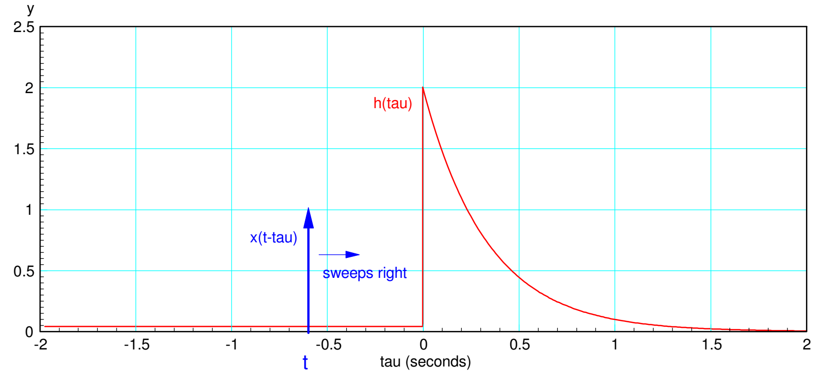
If x(t) is an impulse

$$x(t) = \delta(t)$$

then, graphically, y at any given time, t, is found by

- Plotting the impulse response, $h(\tau)$ (red curve shown above),
- Plotting $x(t-\tau) = \delta(t-\tau)$, and
- Taking the area (integrating) of the product.

y(t) for all time is found by sweeping $x(t-\tau)$ from $-\infty$ to $+\infty$.



The net result will give you h(t), telling you that the impulse response of a filter is the impulse response.

$$y(t) = h(t) = 2e^{-3t}u(t)$$

Next, find y(t) when x(t) is a step input

$$x(t) = u(t)$$

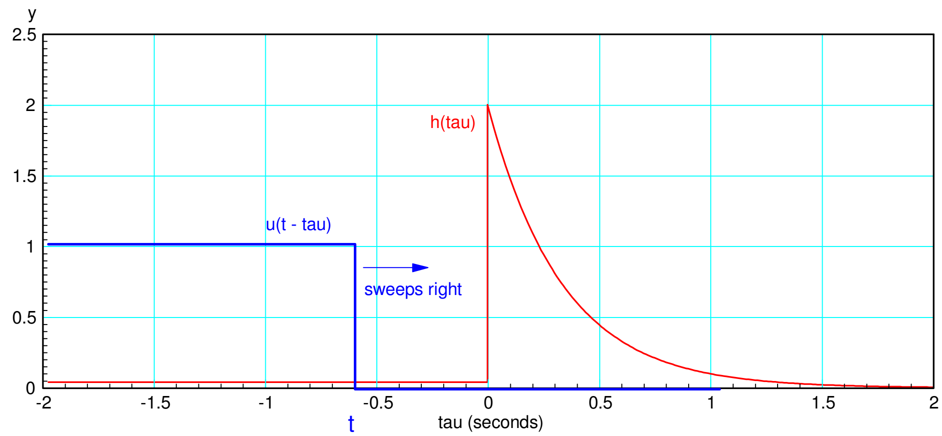
From before, y(t) is from

$$y(t) = \int_{-\infty}^{\infty} x(t-\tau) \cdot h(\tau) \cdot d\tau$$

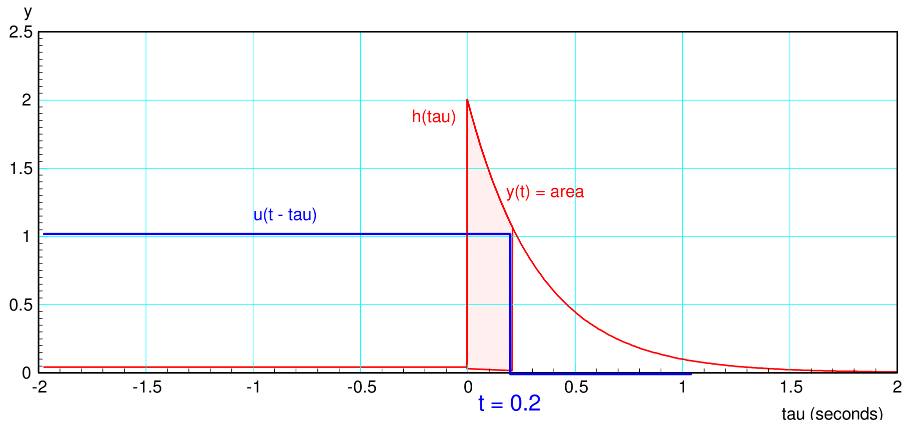
$$y(t) = \int_{-\infty}^{\infty} u(t-\tau) \cdot h(\tau) \cdot d\tau$$

Graphically,

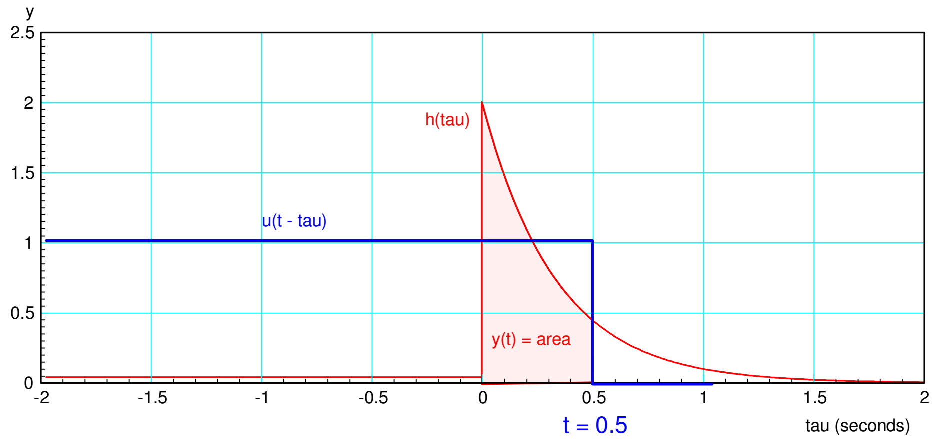
- h(tau) is the impulse reponse
- u(t-tau) is a flipped unit step that sweeps in from the left.



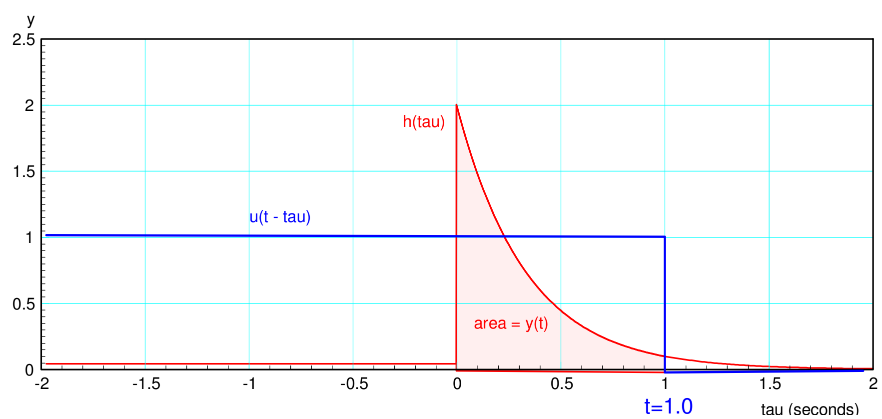
The area under the product of the two functions is y(t). At t = -0.6, y=0 (no overlapping area)



At t=0.2, the common area (y(t)) increases



As time increaes to t = 0.5, the common area increases further (y(t) increases)



As time increases, almost all of the area of h(t) is inclded.

This shows up with the mathematics

Computation of the Convolution Integral:

Find the step response of

$$\frac{dy}{dt} + 3y = 2x$$

for

$$x(t) = u(t)$$

Solution using Convolution: First, find the impulse response

$$h(t) = 2e^{-3t}u(t)$$

Next, evaluate the convolution integral

$$y(t) = \int_{-\infty}^{\infty} x(t-\tau) \cdot h(\tau) \cdot d\tau$$

$$y(t) = \int_{-\infty}^{\infty} u(t-\tau) \cdot 2e^{-3\tau}u(\tau) \cdot d\tau$$

Note that

- $u(\tau) = 0$ for $\tau < 0$

This lets you reset the lower bound on the integal to zero

- $u(t-\tau) = 0$ for $\tau > t$

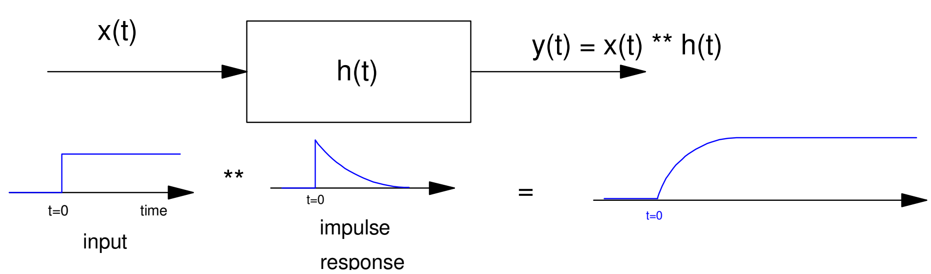
This lets you reset the upper bound on the integral to t

$$y(t) = \int_{0}^{t} 2e^{-3\tau} \cdot d\tau$$

$$y(t) = \left(\frac{-2}{3}e^{-3\tau}\right)_{0}^{t}$$

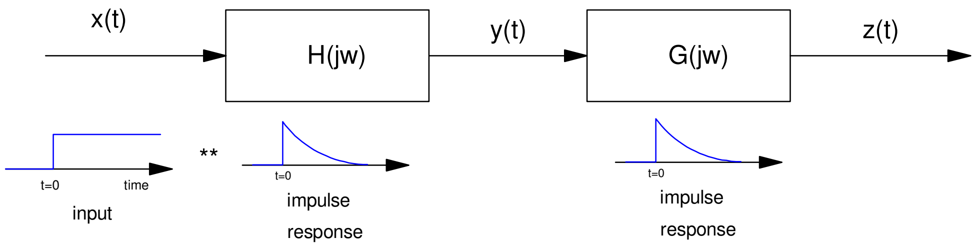
Also note that y(t) = 0 for t<0 (this is a causal system: the output won't happen before the input)

$$y(t) = \left(\frac{2}{3} - \frac{2}{3}e^{-3t}\right)u(t)$$



The output is the convolution of x(t) and h(t): y(t) = x(t) ** h(t)

Convolution (take 2): Suppose you ran the output through another filter:



Find the output of Cascaded Filters

The same result holds: the output of each filter is the convolution of its input with the filter's impulse response

- y(t) = x(t) ** h(t)
- z(t) = y(t) ** g(t)

If the impulse response of G(jw) is

$$g(t) = 4e^{-t}u(t)$$

then

$$z(t) = (4e^{-5t}u(t)) * * \left(\left(\frac{2}{3} - \frac{2}{3}e^{-3t} \right) u(t) \right)$$

$$z(t) = \int_{-\infty}^{\infty} (4e^{-5(t-\tau)}u(t-\tau)) \cdot \left(\left(\frac{2}{3} - \frac{2}{3}e^{-3\tau} \right) u(\tau) \right) \cdot d\tau$$

It's doable - but there has to be a better way...