Natural and Impulse Response

Given a differential equation, such as

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = 4x$$

The natural response is y(t) when

- x(t) = 0
- y has an initial condition at t=0

The impulse response is y(t) when

- y is zero at t=0,
- x(t) = 0 for t>0, and
- x(t) is an impulse at t=0

The response for x(t) not equal to zero comes next lecture.

Natural Response:

Previously, we looked at how to solve differential equations which have periodic inputs. Now let's look at solving differential equations with inputs which are not periodic. The most common inputs are

- No Input: Let the circuit behave naturally with nothing other than its initial conditions.
- An impulse input: This models a car hitting a pot-hole, quickly turning a switch on and off.
- A step response: This models turning on a motor and leaving it on, powering up a DC circuit and leaving it on.

Let's start with the simplest case: solving a differential equation with no input. The only thing powering the circuit are the initial conditions (termed the *natural response.*)

For example, solve the following differential equation

$$\frac{dy}{dt} + 3y = 0$$

with the initial condition

$$y(0) = 5.$$

To solve this, first guess the form of the solution. Assume (i.e. guess) y(t) is of the form

$$y(t) = a \cdot e^{st}$$

Plugging into the differential equation results in

$$s \cdot ae^{st} + 3 \cdot ae^{st} = 0$$

$$(s+3) \cdot ae^{st} = 0$$

Either

- a = 0 (the trivial solution), or
- s = -3

The former doesn't allow us to meet the initial condition, so assume the latter. This means y(t) is of the form

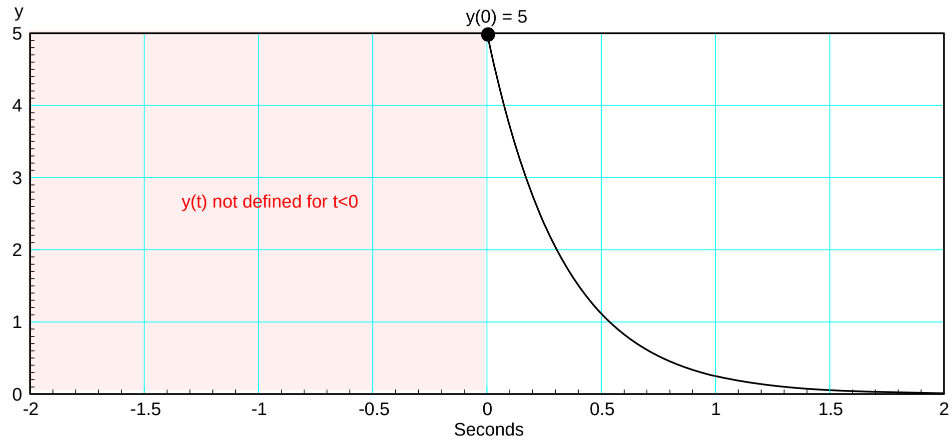
$$y(t) = a \cdot e^{3t}$$

Plugging in s=0

$$y(0) = 5 = a$$

gives us y(t)

$$\boxed{y = 5e^{-3t}}$$



Solution to y(t) given an initial condition of y(0) = 5.

Note that for a 1st-order differential equation, you need one initial condition. For an Nth-order differential equation, you need N initial conditions.

Example 2: Find y(t) assuming y(t) satisfies the following differential equation

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = 0$$

along with the initial conditions

$$y(0) = 5$$

$$\frac{dy(0)}{dt} = 0$$

To solve, again assume y(t) is in the form of

$$y(t) = a \cdot e^{st}$$

Substituting into the differential equation results in

$$s^2 \cdot ae^{st} + 3s \cdot ae^{st} + 2 \cdot ae^{st} = 0$$

$$(s^2+3s+2)\cdot ae^{st}=0$$

Either

- a = 0 (the trivial solution), or
- $s^2+3s+2=0$

The latter has solutions of

s = -1 or -2

meaning

$$y(t)=a\cdot e^{-t}+b\cdot e^{-2t}$$

Plugging in the two initial conditions

$$y(0)=5=a+b$$

$$\dot{y}(0)=0=-a-2b$$

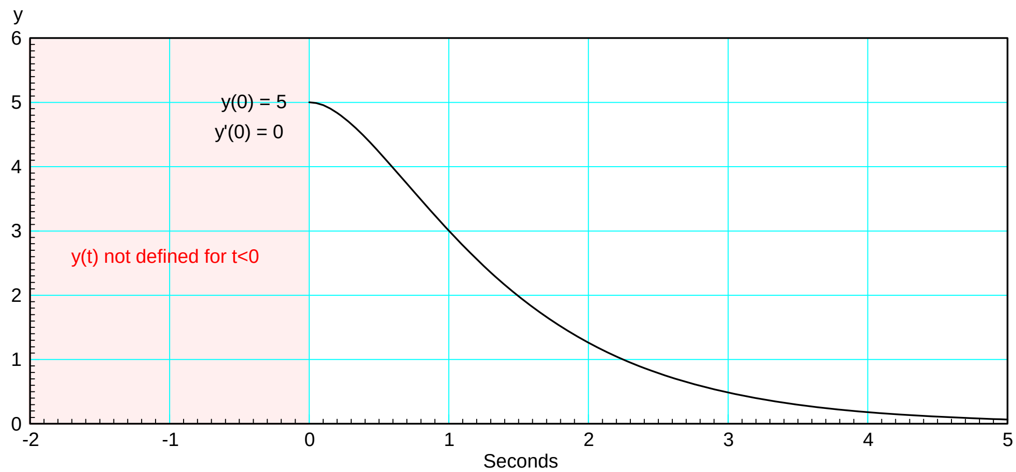
Solving

a = 10

b = -5

so the answer is:

$$\boxed{y(t)=10e^{-t}-5e^{-2t}}$$



Solution to a 2nd-order differential equation with y(0) = 5 and dy/dt = 0 at t=0.

Delta Function: $\delta(t)$

The delta function is a strange function that's really useful. It can be defined as the derivative of a unit step:

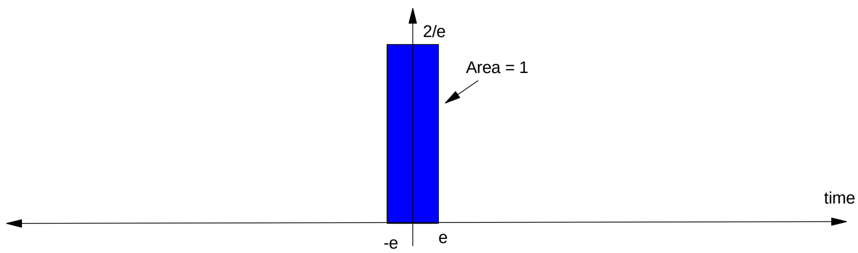
$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$$

$$\delta(t) = \frac{d}{dt}(u(t))$$

Or, it's the function whose integral is the unit step: u(t)

$$\int_{-\infty}^{\infty} \delta(t) dt = u(t)$$

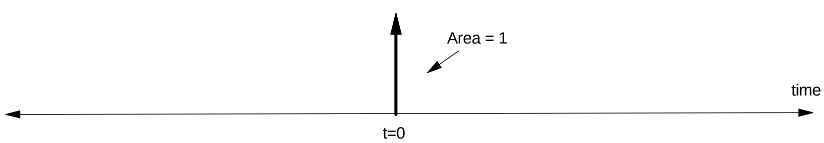
You can also think of the delta function as a small rectangle at t=0 whose area is one.



In the limit as the width goes to zero, the height goes to infinity, and you have a delta function.

$$\delta(t) = \lim_{\varepsilon \to 0} \begin{cases} 0 & t < \varepsilon \\ \frac{2}{\varepsilon} & |t| < \varepsilon \\ 0 & t > \varepsilon \end{cases}$$

which has the symbol of an arrow to indicate the height is infinity and the width is one.



Symbol for $\delta(t)$

The delta function has several useful properties:

y(0): You can determine the initial value of y(t) using a delta function:

$$\int_{-\infty}^{+\infty} y(t) \cdot \delta(t) \cdot dt = y(0)$$

y(t): You can determine the value of y(t) using a delta function:

$$\int_{-\infty}^{+\infty} y(\tau) \cdot \delta(t-\tau) \cdot d\tau = y(t)$$

Note that $\delta(t) = 0$ everywhere except with t=0. Likewise, the only value of y(t) that matters in the above equation is y(T).

Impulse Response

The impulse response to a differential equation is the response when

$$x(t) = \delta(t).$$

For t<0, $\delta(t) = 0$. This results in the initial value for y(t) to also be zero for t<0.

For t>0, $\delta(t) = 0$, resulting in a system which is identical the out previous analysis for finding the natural response of a system.

At t=0, the impulse function sets the initial condition, y(0) and its derivatives.

Example: Find the impulse response of the following system:

$$\frac{dy}{dt} + 3y = 2x$$

$$x(t) = \delta(t)$$

Solution: For t < 0, x(t) =0 and

y(t) = 0 t < 0

For t=0, integrate both sides from t = –ε to t = +ε

$$\int_{-\varepsilon}^{+\varepsilon} \left(\frac{dy}{dt} + 3y\right) dt = \int_{-\varepsilon}^{+\varepsilon} (2\delta(t))dt$$

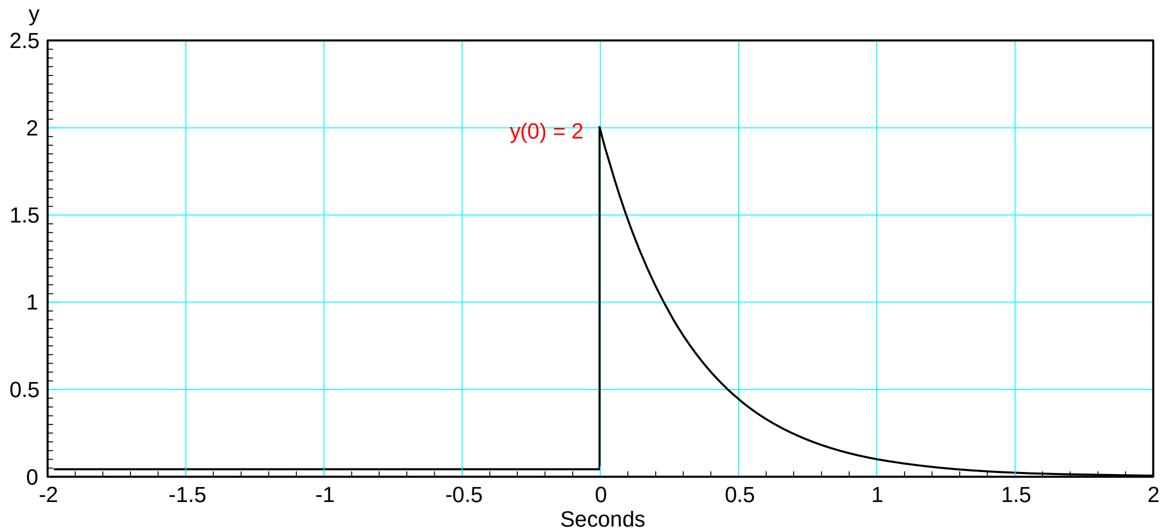
$$y(0) = 2$$

For t>0, x(t) = 0. You're just trying to find the natural response like before. The answer is

$$y(t) = \begin{cases} 0 & t < 0 \\ 2e^{-3t} & t > 0 \end{cases}$$

You can also express this as:

$$y(t) = 2e^{-3t} \cdot u(t)$$



Impulse response, y(t). Note that y(t) is defined for all time

For a 2nd-order differential equation:

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = 4\frac{dx}{dt} + 5x$$

the impulse response is again a natural response with initial conditions. To get the initial conditions, integrate across t=0:

$$\int_{-\varepsilon}^{+\varepsilon} \left(\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y \right) dt = \int_{-\varepsilon}^{+\varepsilon} \left(4\frac{dx}{dt} + 5x \right) dt$$

$$\frac{dy}{dt}(0) + 3y(0) = 4x + 5$$

Integrate again

$$\int_{-\varepsilon}^{+\varepsilon} \left(\frac{dy(0)}{dt} + 3y(0) \right) dt = \int_{-\varepsilon}^{+\varepsilon} (4x + 5) dt$$

$$y(0) = 4$$

$$\frac{dy}{dt}(0) = 5$$

The pattern holds for more and more derivatives. For an nth order differential equation, the coefficients of x tell you the inital conditions: