Circuit Analysis using Fourier Transforms

Background

- If you have a circuit with inductors and/or capacitors, you need to use differential equations to describe that circuit's operation.
- If the input to that circuit is a sinusoid, phasor analysis can be used to analyze that circuit.
- If the input to that circuit is a periodic function which is not a sinusoid, you can use Fourier transforms to convert the input into a sum of sinusoids. Then you can use phasor analysis to evaluate the circuit.

Example 1: Square Wave vs. Sine Wave

A common mistake students make in lab is using a square wave input to measure the gain of a circuit. In a sense, using a square wave for the input is a good thing:

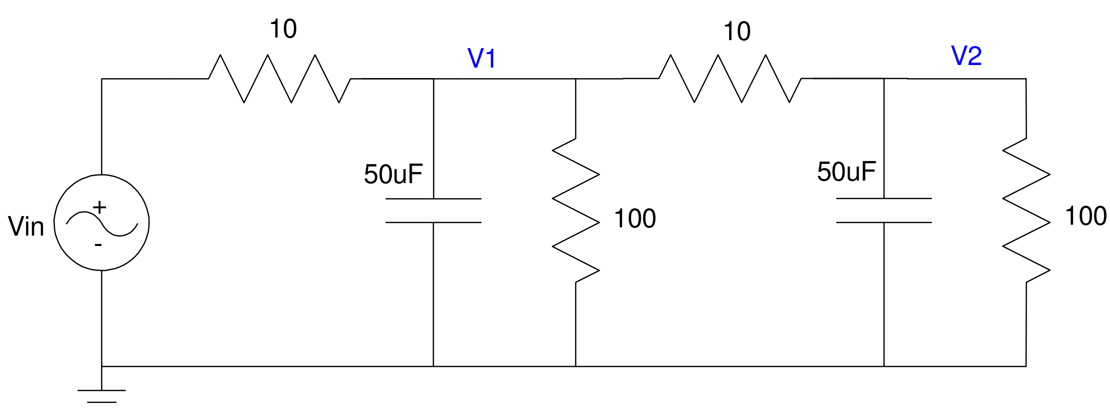
- Square waves contain many frequencies (infinite number of odd harmonics).
- With a single measurement you can determine the gain of a circuit at multiple frequencies.

However, it's also a bad thing

- In order to determine the gain at each frequency, you need to decompose the input and output into their Fourier components.

For example, determine the output, v2(t), when

- Vin is a 10Vpp 100Hz sine wave, and
- Vin is a 10Vpp 100Hz square wave.



Step 1: Find the transfer function from Vin to V2.

The impedance of the capacitors are:

$$Z_c = \frac{1}{j\omega C}$$

Writing the voltage node equations:

$$\left(\frac{V_1 - V_{in}}{10}\right) + \left(\frac{V_1}{1/j\omega C}\right) + \left(\frac{V_1}{100}\right) + \left(\frac{V_1 - V_2}{10}\right) = 0$$

$$\left(\frac{V_2 - V_1}{10}\right) + \left(\frac{V_2}{1/j\omega C}\right) + \left(\frac{V_2}{100}\right) = 0$$

Grouping terms

$$(0.21 + j\omega C)V_1 - (0.1)V_2 = 0.1V_{in} \qquad * \ 0.1$$

$$-0.1V_1 + (0.11 + j\omega C)V_2 = 0 \qquad * \ (0.21 + \text{jwC})$$

Solving

$$(-0.1^2 + (0.11 + j\omega C)(0.21 + j\omega C))V_2 = 0.01V_{in}$$

$$\left((j\omega C)^2 + 0.32(j\omega C) + 0.0131\right)V_2 = 0.01V_{in}$$

$$V_2 = \left(\frac{0.01}{(j\omega C)^2 + 0.32(j\omega C) + 0.0131}\right)V_{in}$$

Step 2: Determine Vin in terms of sinusoids. If Vin is a 10Vpp 100Hz square wave

$$v_{in(t)} = \begin{cases} +5V & \sin(200\pi t) > 0 \\ -5V & \sin(200\pi t) < 0 \end{cases}$$

then take the Fourier transform for Vin. From before, a 1Vpp (0-1V) square has the Fourier transform of

$$V_{in} = \sum_{\text{n odd}} \left(\frac{2}{n\pi}\right) \sin(n\omega_0)$$

A 10Vpp square wave is therefore

$$V_{in} = \sum_{\text{n odd}} \left(\frac{20}{n\pi}\right) \sin(n\omega_0 t)$$

$$v_{in} = 6.36 \sin(628t) + 2.12 \sin(1884t) + 1.27 \sin(3140t) + \ldots$$

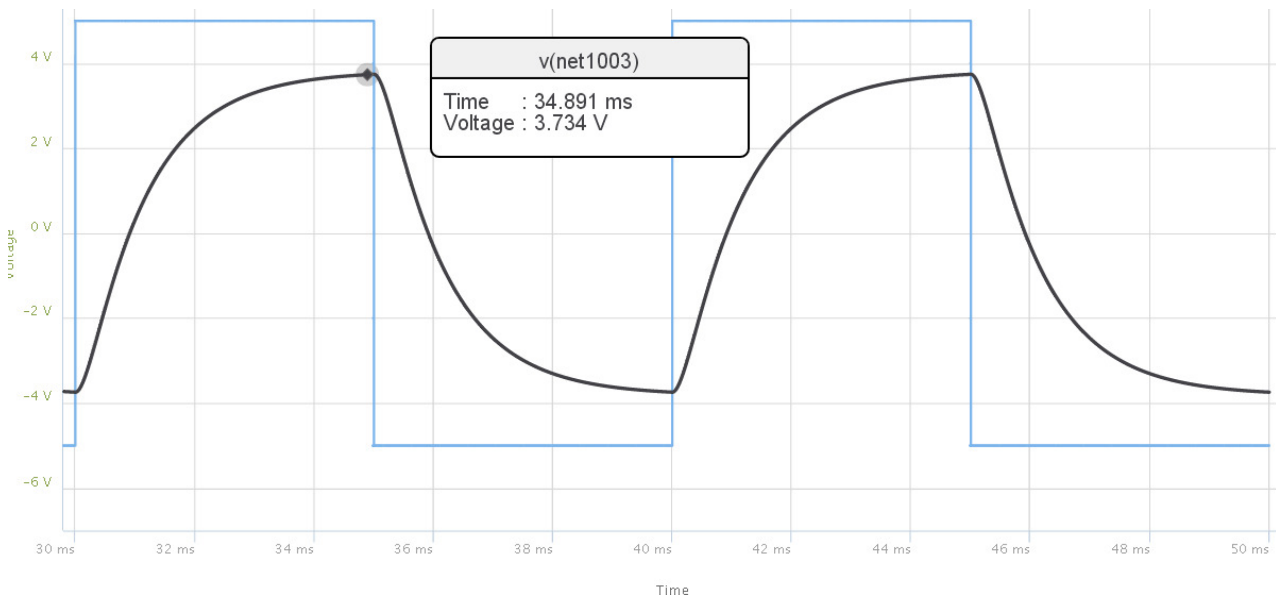
Step 3: Find V2(t). Using superposition, treat this as three separate problems:

100Hz	300Hz	500Hz
input:	input:	input:
$v_{in} = 6.26 \sin(628t)$	$v_{in} = 2.12 \sin(1884t)$	$v_{in} = 1.27 \sin(3140t)$
gain:	gain:	gain:
$G(j\omega) = 0.635\angle -40^0$	$G(j\omega) = 0.328\angle -82^0$	$G(j\omega) = 0.194\angle -103^0$
output = gain * input	output = gain * input	output = gain * input
$v_2 = 3.98 \sin(628t - 40^0)$	$+0.70 \sin(1884t - 82^0)$	$+0.25 \sin(3140t - 103^0)$

The net result is

$$v_2(t) = 3.98 \sin(628t - 40^0) + 0.70 \sin(1884t - 82^0) + +0.25 \sin(3140t - 103^0)$$

Checking in PartSim:



Output for the 2-stage RC filter with a 100Hz, 10Vpp square wave input

Note that the gain at 100Hz isn't easy to tell form this result: the output contains several frequencies.

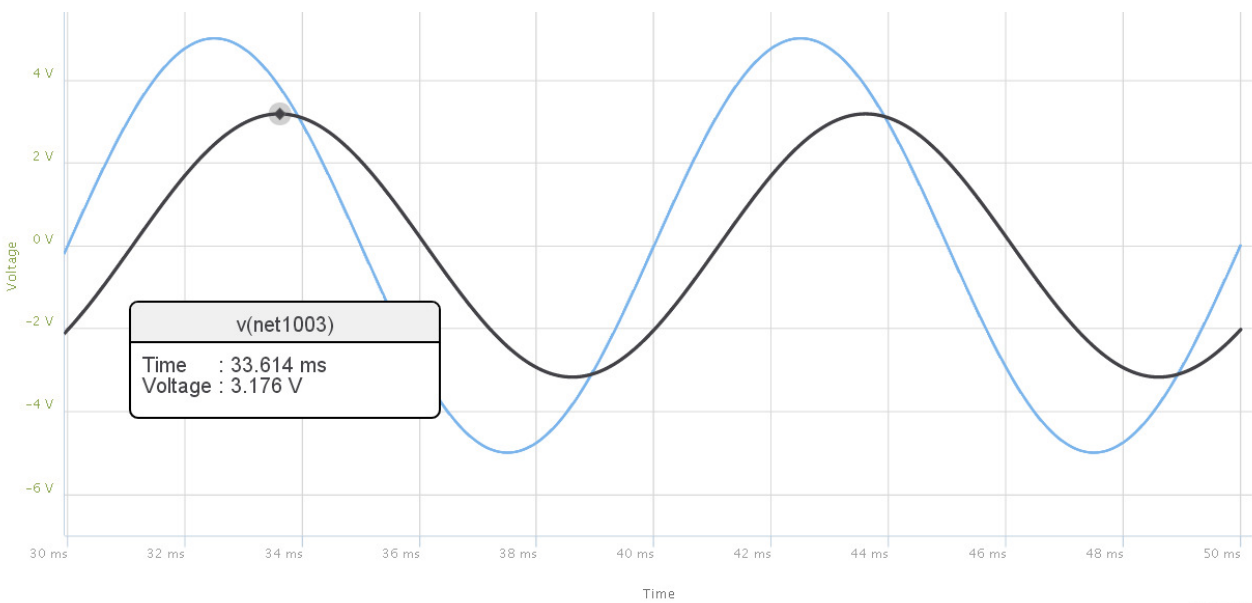
If you *did* take the Fourier transform of the input and output, you could determine the gain at several frequencies all at once. That takes a lot of work though...

If instead you apply a 10Vpp 100Hz sine wave, then the output is

100Hz	300Hz	500Hz
input: $v_{in} = 5\sin(628t)$	input: $v_{in} = 0$	input: $v_{in} = 0$
gain: $G(j\omega) = 0.635\angle -40^0$	gain: *doesn't matter: input is zero.*	gain: *doesn't matter: input is zero.*
output = gain * input $v_2 = 3.18\sin(628t - 40^0$	output = gain * input +0	output = gain * input +0

$$\boxed{v_2 = 3.18\sin(628t - 40^0}$$

Checking in PartSim:



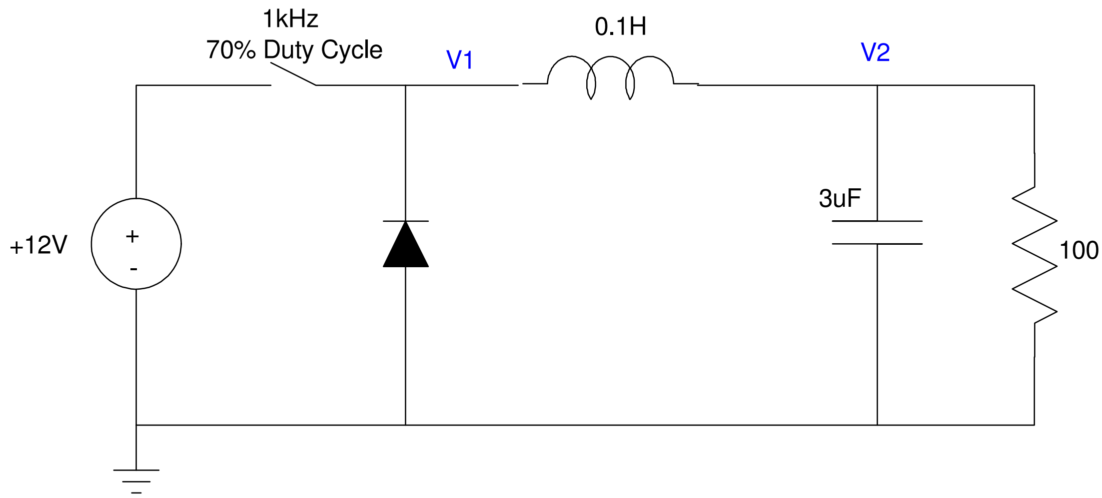
Output for the 2-stage RC filter with a 100Hz, 10Vpp sine wave input

Note that

- $gain = \left(\frac{3.176V}{5V}\right) = 0.634$ - which matches up with the theoretical gain of 0.635
- $phase = -\left(\frac{\text{1.1ms delay in output}}{\text{10ms period}}\right) \cdot 360^0 = -39.6^0$ - which matches up with the theoretical phase shift of -40 degrees.

Example 2: Buck Converter.

This circuit converts a 12VDC power supply to a lower voltage at V2 with a small ripple. Find the voltage at V2:

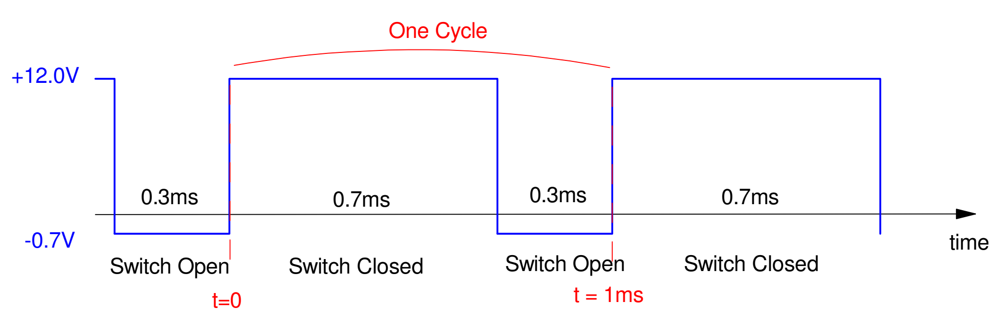


To analysis this circuit in ECE 320, we change the problem so that it's easier to solve but keeps the flavor of the original problem. In ECE 320, we assume that V1 has two terms:

- A DC term (8.19V) which is the average of 12V (when the switch is closed) and -0.7V (when the switch is open), and
- An AC term (12.7Vpp, 1kHz).

The answer you get is close, but slightly different that PartSim gives you. A more accurate answer requires Fourier transforms.

Step 1: Express V1(t) in terms of its Fourier series. V1(t) looks like the following:



The DC level is the average voltage:

$$V_1(0) = 0.7 \cdot 12V + 0.3 \cdot (-0.7V)$$

$$V_1(0) = 8.190V$$

The fundamental frequency is

$$\omega_0 = \frac{2\pi}{T} = 2000\pi \text{ rad/sec}$$

Taking the derivative gives delta functions:

$$\frac{dv_1}{dt} = 12.7\delta(t) - 12.7\delta(t - 0.7ms)$$

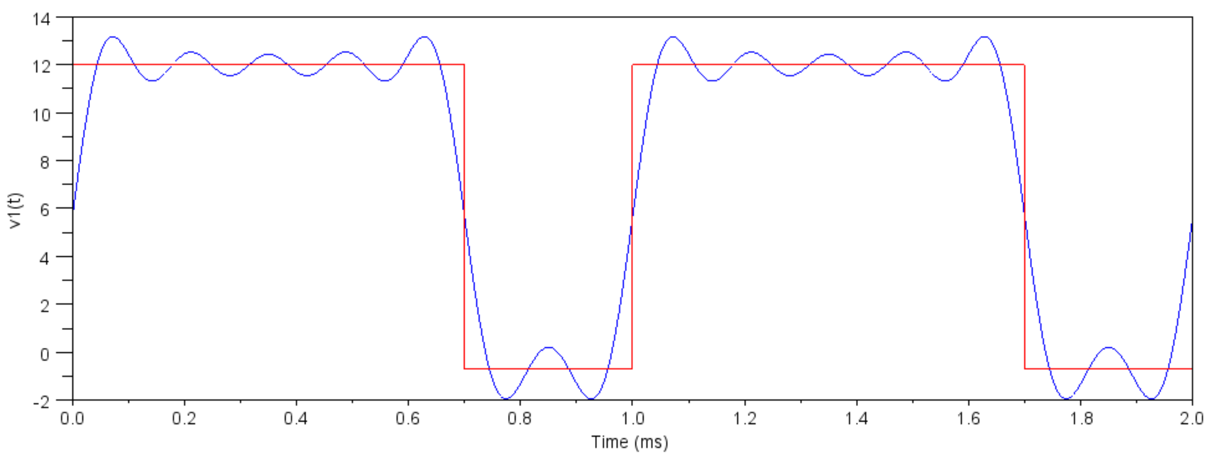
This gives the complex Fourier transform of

$$(jn)V_1 = \left(\frac{12.7}{2\pi}\right)(1 - e^{-jn \cdot 1.4\pi})$$

$$V_1(n\omega_0) = \left(\frac{-j6.35}{\pi n}\right)(1 - e^{-j1.4n\pi})$$

Checking in Matlab:

```
DC = 12*0.7 - 0.3*0.7

    8.19

n = [1:7]';

V1 = -j*6.35 ./ (pi * n) .* (1 - exp(-j*1.4*n*pi));

v1 = DC + 0*t;

for n=1:7
  v1 = v1 + 2*real(V1(n))*cos(n*w0*t) - 2*imag(V1(n))*sin(n*w0*t);
  end
```



Step 2: Use phasor analysis to find the transfer function from V1 to V2

$$L \rightarrow j\omega L$$

$$C \rightarrow \frac{1}{j\omega C}$$

Writing the voltage node equation at V2:

$$\left(\frac{V_2-V_1}{j\omega L}\right)+\left(\frac{V_2}{\left(\frac{1}{j\omega C}\right)}\right)+\left(\frac{V_2}{R}\right)=0$$

$$\left(\frac{V_2-V_1}{j\omega L}\right)+((j\omega C)V_2)+\left(\frac{V_2}{R}\right)=0$$

$$\left(\left(\frac{1}{j\omega L}\right)+(j\omega C)+\left(\frac{1}{R}\right)\right)V_2=\left(\frac{1}{j\omega L}\right)V_1$$

$$V_2=\left(\frac{\left(\frac{1}{j\omega L}\right)}{\left(\left(\frac{1}{j\omega L}\right)+(j\omega C)+\left(\frac{1}{R}\right)\right)}\right)V_1$$

With a little algebra

$$V_2=\left(\frac{R}{(j\omega)^2RLC+j\omega L+R}\right)V_1$$

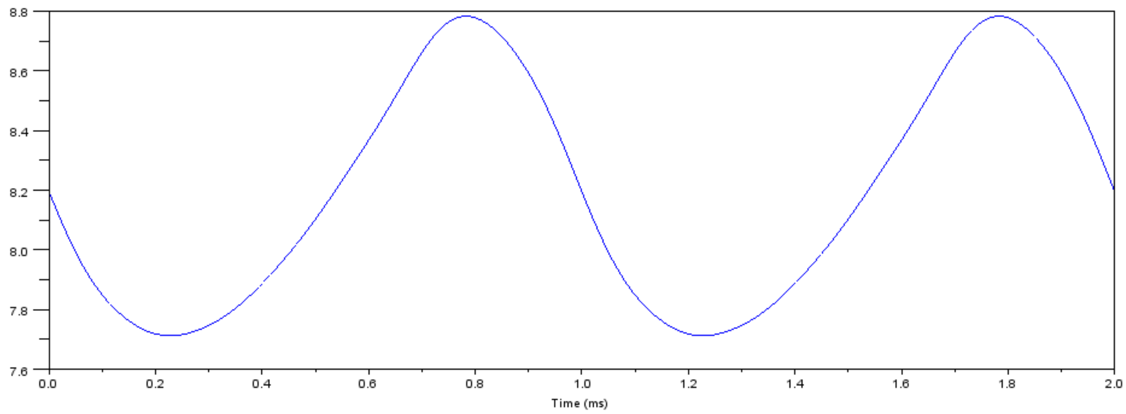
Step 3: Compute V2. Output is gain times input:

Harmonic	Frequency	V1(n)	$G(j\omega)$	Yn
n	$j\omega = jn\omega_0$	$\left(\frac{-j3.175}{\pi^2 n}\right)(1-e^{-j1.4n})$	$\left(\frac{R}{(j\omega)^2RLC+j\omega L+R}\right)$	$G(j\omega)\cdot X_n$
0	0	8.190	1.000	8.190
1	j 6,283	- 1.9223 - 2.6459i	- 0.069 - 0.04i	0.0269 + 0.2596i
2	j 12,566	0.594 - 1.8283i	- 0.0201 - 0.0054i	- 0.0219 + 0.0335i
3	j 18,850	0.396 - 0.1287i	- 0.0092 - 0.0016i	- 0.0038 + 0.0005i
4	j 25,133	- 0.4806 - 0.3492i	- 0.0052 - 0.0007i	0.0023 + 0.0022i
5	j 31,416	- 0.8085i	- 0.0034 - 0.0004i	- 0.0003 + 0.0027i
6	j 37,699	0.3204 - 0.2328i	- 0.0023 - 0.0002i	- 0.0008 + 0.0005i
7	j 43,982	- 0.1697 - 0.0551i	- 0.0017 - 0.0001i	0.0003 + 0.0001i

In Matlab:

```
n = [1:7]';
w0 = 2*pi/T;
s = j*n*w0;

G = R ./ (R*L*C*(s.^2) + L*s + R);

V2 = G .* V1;

v2 = DC + 0*t;

for n=1:7
```

```
    v2 = v2 + 2*real(V2(n))*cos(n*w0*t) - 2*imag(V2(n))*sin(n*w0*t);
    end

plot(t*1000,v2)
xlabel('Time (ms)');
```



Comparing this answer to what we compute in ECE 320:

The actual peak-to-peak voltage at V2 is

```
V2pp = max(v2) - min(v2)

    1.0700211
```

Using only the 1st harmonic and assuming V1pp(n=1) is 12.7Vpp results in

```
V2pp_approx = abs( G(1) * 12.7 )

    1.0133753
```