# Complex Numbers and Phasors

Back in Calculus, the method used to solve differential equations with a forcing function was to

- Assume the form of the solution (usually similar to the input),
- Plug into the differential equation, then
- Solve

For exponential inputs, this results in solving 1 equation for 1 unknown. For sinusoidal inputs, this resulted in solving 2 equations for 2 unknowns (one for sine, one for cosine).

A similar pattern follows if you try to write and solve the voltages for a circuit with N voltage nodes:

- If the input is an exponential, you'll solve N equations for N unknowns (one equation for each node)
- If the input is a sinusoid, you'll have to solve 2N equations for 2N unknowns (N nodes, each with 2 unknowns: one for sine, one for cosine).

This is a problem. It would be a *lot* easier if you could just solve N equations for N unknowns.

Phasors let you do that.

The idea behind phasors is as follows:

- I like exponential inputs. If the input is an exponential, the differential equation is easy to solve.
- Therefore, if you have a sinusoidal input, change the problem. Make it an exponential input.

This is a common engineering trick: if you have a difficult problem to solve, change it so that it's easy to solve. Just change it so that you keep the flavor of the original problem.

In order to use phasors, you first have to use complex numbers.

## Complex Numbers

Let

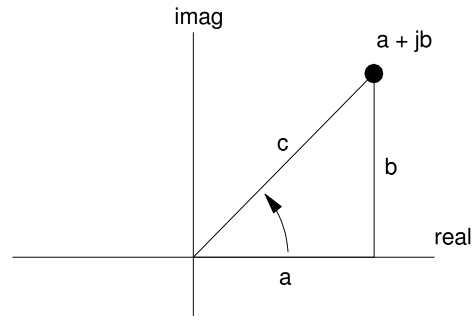
$$j = \sqrt{-1}$$

Any given number can then have a real and a complex part

$$x = a + jb$$

You can express this number in rectangular form ( a + jb) or polar form

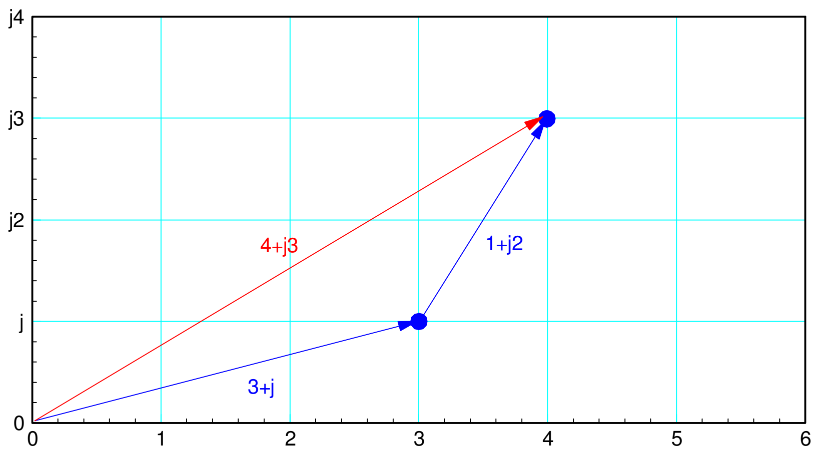
$$x = c\angle\theta$$




Complex Number (a + jb) can also be expressed as $(c\angle\theta)$

When you add complex numbers, the rectangular form is more convenient: the real part adds and the complex part adds.

$$(a_1 + jb_1) + (a_2 + jb_2) = (a_1 + a_2) + j(b_1 + b_2)$$




When you add complex numbers, the real part adds and the complex part adds

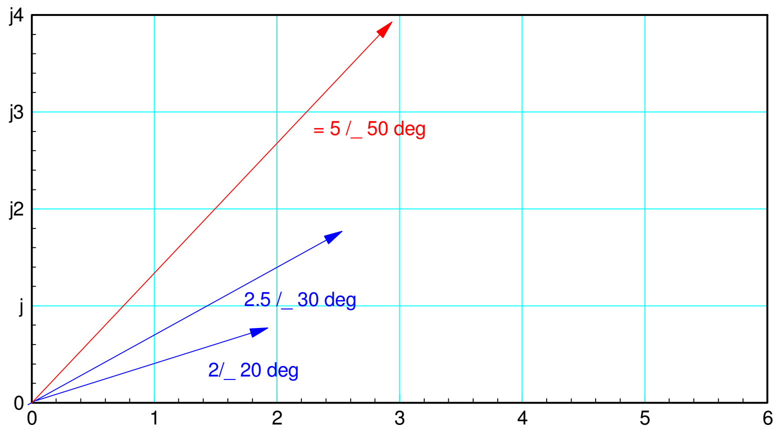
When you multiply complex numbers, polar form is more convenient: the amplitudes multiply, the angles add:

$$(c_1\angle\theta_1)(c_2\angle\theta_2) = c_1c_2\angle(\theta_1 + \theta_2)$$

It also works in rectangular form

$$(a_1 + jb_1)(a_2 + jb_2) = a_1a_2 + ja_1b_2 + jb_1a_2 + j^2b_1b_2$$

$$= (a_1a_2 - b_1b_2) + j(a_1b_2 + a_2b_1)$$




When you multiply complex numbers, the amplitude multiplies (2*2.5=5) and the angle adds (30+30=50)

The complex conjugate of a complex number is

$$(a + jb)^* = a - jb$$

A property of complex conjugates is that a number times its complex conjugate is the amplitude squared

$$(a + jb)(a - jb) = a^2 + b^2$$

(the complex portion cancels).

Dividing complex numbers is easiest to do in Matlab or with a calculator. You can do it by hand, however. If

$$X = \frac{a+jb}{c+jd}$$

then clear the denominator by multiplying top and bottom by the complex conjugate:

$$X = \left(\frac{a+jb}{c+jd}\right) \cdot \left(\frac{c-jd}{c-jd}\right)$$

$$X = \left(\frac{(ac+bd)+j(bc-ad)}{c^2+d^2}\right)$$

$$X = \left(\frac{ac+bd}{c^2+d^2}\right) + j\left(\frac{bc-ad}{c^2+d^2}\right)$$

You can evaluate functions of complex numbers. For example,

$$G(s) = \left(\frac{1000}{s(s+5)(s+20)}\right)_{s=j5}$$

is

$$= \left(\frac{1000}{(j5)(5+j5)(20+j5)}\right)$$

$$= -1.1764 - j0.7058$$

In Matlab

```
--> s = j*5;
-->Gs = 1000 /  ( s * (s+5) * (s+20) )

  - 1.1764706 - 0.7058824i
```

(note:  You should have a hand calculator which can do this.  You'll be using complex  numbers extensively in Electrical and Computer Engineering.  I'd personally recommend an HP35s.)

You can also solve N equations for N unknowns with complex numbers.  For example, solve

$$\begin{bmatrix} 10 & j3 & 5+j7 \\ 2 & 15 & 3-j \\ 6 & 8 & j6 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 1 \\ j2 \\ 3+j4 \end{bmatrix}$$

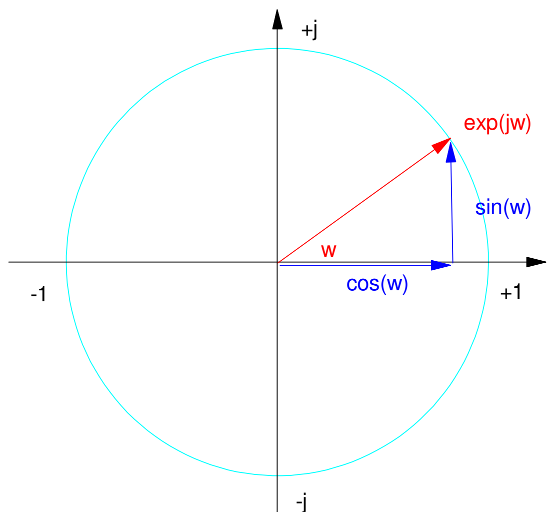
In Matlab:

```
-->A = [10, j*3, 5+j*7 ; 2, 15, 3-j ; 6, 8, j*6]

    10.     3.i     5. + 7.i
    2.      15.     3. - i
    6.      8.      6.i

-->B = [1 ; j*2 ; 3+j*4]

    1.
    2.i
    3. + 4.i

-->V = inv(A)*B

  - 0.3555559 + 0.3211028i
    0.0910127 + 0.2391413i
    0.0267087 - 0.7342056i
```

## Phasors

The basic identity behind phasors is the complex exponential (Euler's identity)

$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$$




Euler's Identity: $e^{j\omega} = \cos(\omega) + j\sin(\omega)$

This allows us to represent a cosine wave

$$x(t) = \cos(\omega t)$$

as the real part of a complex exponential

$$x(t) = real(e^{j\omega t})$$

With phasors, the real() is assumed: we represent a cosine wave as

$$x(t) = e^{j\omega t}$$

To find the actual input, take the real part of x(t). To find the actual output, take the real part of y(t).

If the input is a combination of sine and cosine terms, use a complex numer

$$(a + jb) \cdot e^{j\omega t} = (a + jb) \cdot (\cos(\omega t) + j\sin(\omega t))$$

$$= (a\cos(\omega t) - b\sin(\omega t)) + j(a\sin(\omega t) + b\cos(\omega t))$$

Taking the real part

$$real[(a + jb) \cdot e^{j\omega t}] = a\cos(\omega t) - b\sin(\omega t)$$

**With phasors:**

**The real part of a+jb represents cos(wt)**

**The complex part of a+jb represents -sin(wt)**

$$a + jb \rightarrow a\cos(\omega t) - b\sin(\omega t)$$

or if you prefer polar form

$$a\angle\theta \rightarrow a\cos(\omega t + \theta)$$

Notation: The standard with circuit analysis is to use

- Lower case letters to represent x(t)
- Capital letters to represent the phasor representation for x(t)

For example, if

$$x(t) = 2\cos(3t) + 4\sin(3t)$$

then

$$X = 2 - j4$$

Also note that if x(t) and y(t) are in the form of

$$y(t) = a \cdot e^{j\omega t}$$

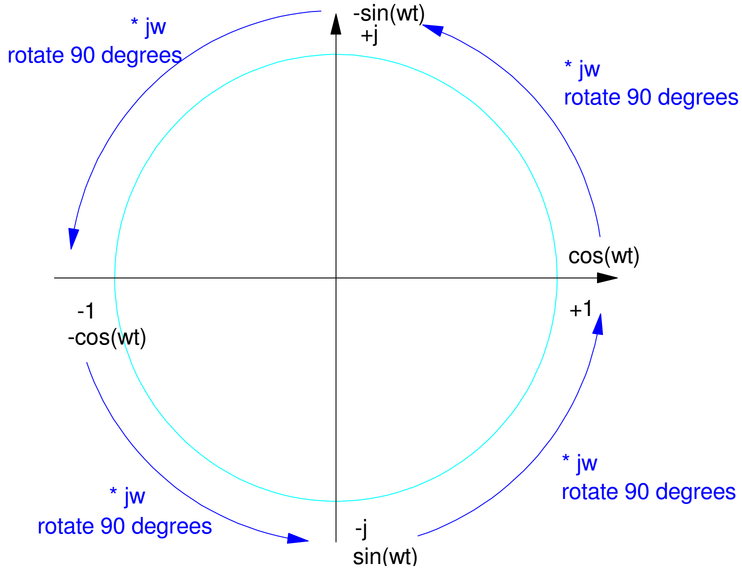
where 'a' is a complex number (real for cosine, imaginary for sine), then

$$\frac{dy}{dt} = j\omega \cdot ae^{j\omega t}$$

You can replace derivatives with 'jw'. This turns differential equations into algebraic equations - with the assumption that algebra is easier than calculus. Solving an Nth-order differential equation then becomes solving a single equation for one unknown - albeit a complex number for that unknown.

Note that equating differentiation with multiplying by 'jw' works:

| time | Phasors (w = 1) |
|---|---|
| $\frac{d}{dt}(\cos(t)) = -\sin(t)$ | $j\omega \cdot 1 = j$ |
| $\frac{d}{dt}(-\sin(t)) = -\cos(t)$ | $(j\omega) \cdot j = -1$ |
| $\frac{d}{dt}(-\cos(t)) = \sin(t)$ | $(j\omega) \cdot (-1) = -j$ |
| $\frac{d}{dt}(\sin(t)) = \cos(t)$ | $(j\omega) \cdot (-j) = +1$ |




Differentiation is equivalent to multiplying by jw ( times w, rotate +90 degrees)

This allows you to solve differential equations with sinusoidal inputs. For example, find y(t)

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = 4x$$

when

$$x(t) = 3\sin(5t)$$

First, convert to phasors.

$$X = 0 - j3$$

$$\omega = 5$$

Substituting

$$\frac{d}{dt} \rightarrow j\omega$$

results in the differential equation becoming

$$(j5)^2 Y + 3(j5)Y + 2Y = 4(0 - j3)$$

$$Y = \left(\frac{-j12}{-23+j15}\right)$$

$$Y = -0.2387 + j0.366$$

which is phasor notation for

$$y(t) = -0.2387\cos(5t) - 0.366\sin(5t)$$

or, if you prefer polar form,

$$Y = 0.4370\angle 123^0$$

which is phasor notation for

$$y(t) = 0.4370\cos(5t + 123^0)$$

## Transfer Functions with Phasors

Given a system which is described by a differential equation, such as

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = 4x$$

if you assume that x(t) is a sinusoid, then you can replace differentiation with multiplication by 'jw'

$$(j\omega)^2 Y + 3(j\omega)Y + 2Y = 4X$$

Solving for Y

$$Y = \left(\frac{4}{(j\omega)^2 + 3j\omega + 2}\right)X$$

or

$$Y = G(j\omega)X$$

Here, G(jw) is called *the transfer function from X to Y*. This is a more convenient way of writing the general solution, since it applies to any frequency. For example, if

$$x(t) = 4\sin(5t)$$

then the phasor representation for x is

$$X = 0 - j4$$

(zero cosine, 4 sine). The frequency is

$$\omega = 5 \text{ rad/sec.}$$

The gain at this frequency is

$$\left(\frac{4}{(j\omega)^2 + 3j\omega + 2}\right)_{\omega=5} = -0.1220 - j0.0796$$

The output is then

$$Y = G \cdot X$$

$$Y = (-0.1220 - j0.0796) \cdot (0 - j4)$$

$$Y = -0.3183 + j0.4881$$

$$y(t) = -0.3183\cos(5t) - 0.4881\sin(5t)$$

If you change the input to

$$x(t) = 6\cos(7t)$$

then in phasor form

$$Y = G \cdot X$$

$$Y = \left(\frac{4}{(j\omega)^2 + 3j\omega + 2}\right)_{\omega=7} \cdot (6 + j0)$$

$$Y = (-0.0709 - j0.031) \cdot (6 + j0)$$

$$Y = -0.4257 - j0.1902$$ rectangular form

$$Y = 0.4663\angle - 156^0$$ polar form

meaning

$$y(t) = -0.4257\cos(7t) + 0.1902\sin(7t)$$

$$y(t) = 0.4663\cos\left(7t - 156^0\right)$$

(both answers are correct)