Butterworth, Elliptic, Chebychev Filters

Objective:

- Know what each filter tries to optimize
- Know how these filters compare

An Nth-order RC filter is the closest appoximation to an ideal low pass filter subject to

- There are N poles
- All poles are real
- There are no zeros
- The maximum gain is 1.000

An Nth-order Butterworth filter is the closest appoximation to an ideal low pass filter subject to

- There are N poles
- There are no zeros
- The maximum gain cannot exceed 1.0000

An Nth-order Type-1 Chebychev filter is the closest approximation to an ideal low pass filter subject to

- There are N poles
- There are no zeros
- The maximum gain cannot exceed $1 + \varepsilon$. (Some ripple is permitted).

An Nth-order Type-2 Chebychev filter is the closest approximation to an ideal low pass filter subject to

- There are N poles
- There are N zeros
- The maximum gain cannot exceed $1 + \varepsilon_1$. (Some ripple is permitted).
- The maximum gain the band reject region cannot exceed ε_2

RC Filter

An n-pole RC filter has all n-poles on the real axis.
- It's advantage is you can build it with a passive RC filter (good)
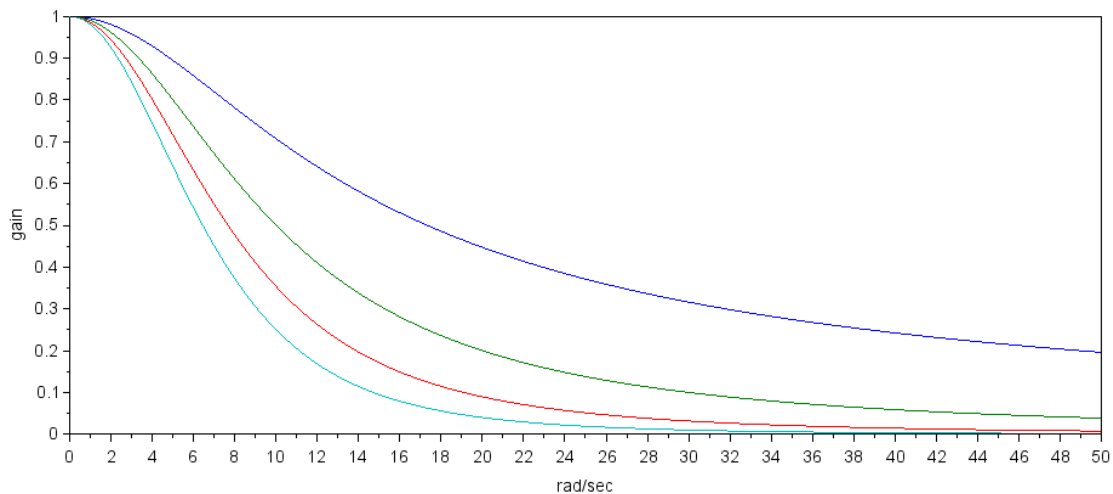- It's problem is it's a pretty poor filter.

For example, the gain of

$$G(s) = \left(\frac{10}{s+10} \right)^n$$

for n=1, 2, 3, and 4 is shown below. Note that
- As n increases, the high-frequency gain gets smaller and smaller (good)
- However, the gain below 5 rad/sec starts to droop more an more (bad)

```
-->w = [0:0.1:50]';
-->s = j*w;
-->G = 10 ./ (s+10);
-->plot(w,abs([G,G.^2,G.^3,G.^4]));
-->xlabel('rad/sec');
-->ylabel('gain');
```

Gain of and RC filter for n=1 (blue) to n=4 (cyan)

Butterworth Filter:

If you use complex poles, you can do better. For example, take the case of n=5.

A 5th-order RC filter with a corner at 10 rad/sec is

$$G = \left(\frac{10}{s+10} \right)^5$$

One of these poles has to be real. The other four, however, could be moved along the circle centered at the origin with a radius of 10. As you increase the angle of these poles, the gain at j10 increases. If you go too far, the gain at j10 starts to go above one.

A Butterworth filter is the farthest you can slide the poles while keeping the maximum gain less than one

As it turns out, the optimal filter (i.e. the closest to an ideal low-pass filter) with

- N poles
- No zeros, and
- No resonance (max gain <= 1.0000)

has N poles with

- The amplitude of the poles is equal to the corner frequency, and
- The angle between the poles is

$$\phi = \frac{180^0}{N}$$

If the corner is 1 rad/sec, for example, the poles for a Butterworth filter are

N	2	3	4	5
(s+a)	$1\angle \pm 45^0$	1 $1\angle \pm 60^0$	$1\angle \pm 22.5^0$ $1\angle \pm 67.5^0$	-1 $-1\angle \pm 36^0$ $-1\angle \pm 72^0$

Example: Design a 4th-order Butterworth low-pass filter with a corner at 10 rad/sec.
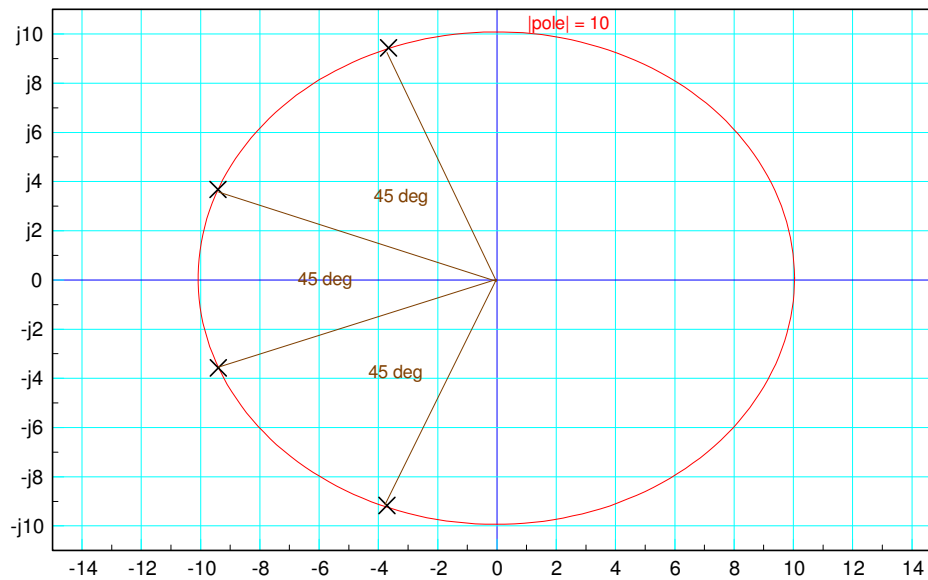
Solution: The poles are 45 degrees apart

$$\phi = \frac{180^0}{4} = 45^0$$

The amplitude of the poles is the corner frequency (10 rad/sec). Hence

$$G(s) = \left(\frac{10^4}{\left(s+10\angle 22.5^0\right)\left(s+10\angle -22.5^0\right)\left(s+10\angle 67.5^0\right)\left(s+10\angle -67.5^0\right)} \right)$$

or since complex poles are always in complex conjugate pairs,

$$G(s) = \left(\frac{10^4}{\left(s+10\angle\pm22.5^0\right)\left(s+10\angle\pm67.5^0\right)} \right)$$

The resulting gain vs. frequency is then

```
p1 = 10 * exp(j*22.5*pi/180);
p2 = conj(p1);
p3 = 10 * exp(j*67.5*pi/180);
p4 = conj(p3);

w = [0:0.01:30]';
s = j*w;

Gd = 1 * (w < 10);

Gs = 10^4  ./ ( (s+p1) .* (s+p2) .* (s+p3) .* (s+p4) );

plot(w,abs(Gs),'b',w,abs(Gd),'m');
xlabel('Frequency (rad/sec)');
ylabel('Gain');
```
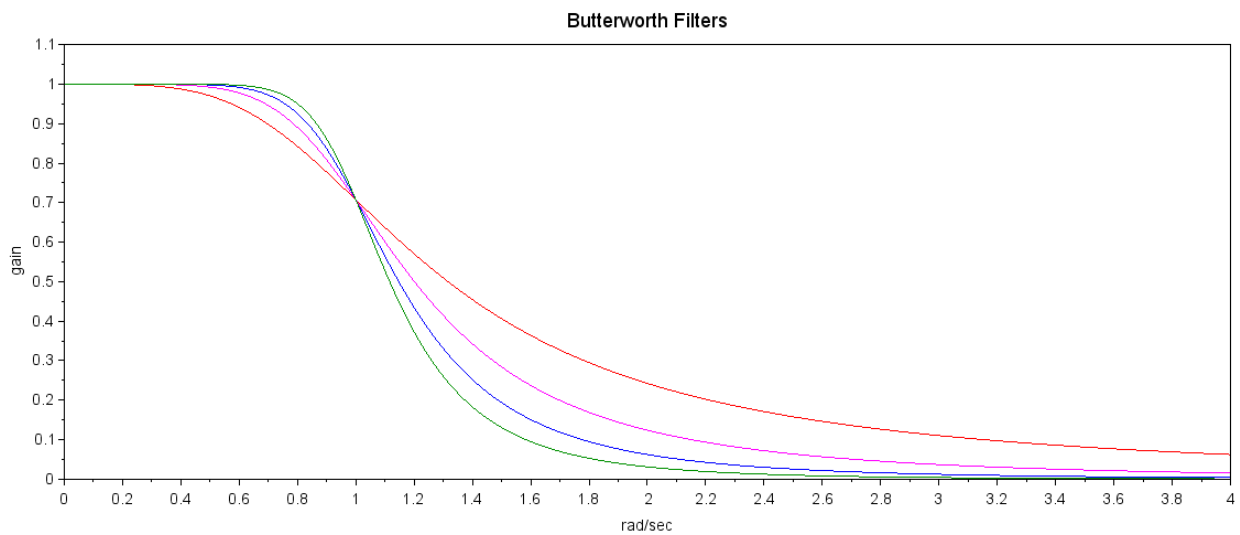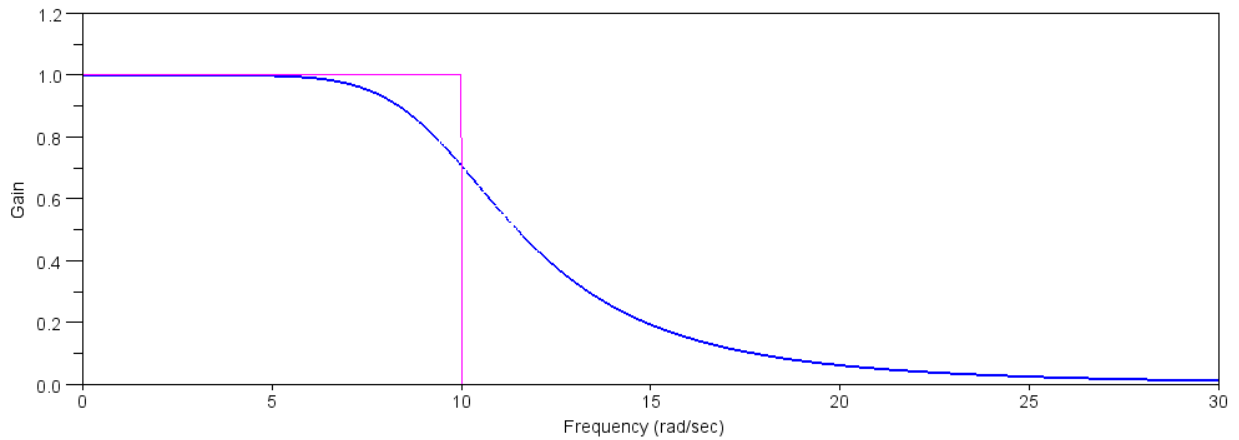
Gain of a Butterworth Filter for n=2 (red), 3 (magenta), 4 (blue), 5 (green)

Note that with a Butteworth filter,

- The more poles you have the closer it gets to an ideal low-pass filter.
- All poles have the same amplitude
- All poles have an equal spacing between them

Type-1 Chebychev Filter

A Chebuchev filter is similar to a Butterworth filter, except you allow the gain to be slightly larger than one at some point. In return, you get a filter which is closer to ideal.

Unlike a Butterworth filter, there are an infinite number of Chebychev filters: one for each resonance you allow. The poles for a Type-1 Chebychev filter with a corner at 1 rad/sec follow and 0.2 ripple are given below.

	N=2	N=3	N=4	N=5	N=6
zeros	none	none	none	none	none
poles	$-1.60\angle \pm 50.7^0$	-0.85 $-1.21\angle \pm 69.5^0$	$-0.72\angle \pm 38.5^0$ $-1.11\angle \pm 77.8^0$	-0.48 $-0.76\angle \pm 59.3^0$ $-1.06\angle \pm 82.0^0$	$-0.47\angle \pm 36.1^0$ $-0.81\angle \pm 69.8^0$ $-1.04\angle \pm 84.4^0$

n = 2		n = 3		n = 4	
| pole |	degrees	| pole |	degrees	| pole |	degrees
1.5967184	50.654966	1.2130758	69.505382	1.1075629	77.815449
		0.8494427		0.7208251	38.470416

n = 5		n = 6		n = 7	
| pole |	degrees	| pole |	degrees	| pole |	degrees
1.0649391	82.007862	1.0435768	84.37776	1.0313304	85.837824
0.7583392	59.258028	0.8099629	69.827002	0.8511278	75.734189
0.4791522		0.4722575	36.104208	0.5490110	56.492848
				0.3363897	

n = 8		n = 9		n = 10	
| pole |	degrees	| pole |	degrees	| pole |	degrees
1.0236433	86.797811	1.0184934	87.461528	1.0148704	87.939028
0.8816164	79.356269	0.9041474	81.740491	0.9210463	83.396355
0.6281433	67.171719	0.6932953	73.319611	0.7446030	77.199244
0.3520881	35.268226	0.4294884	55.363465	0.5104322	65.966884
		0.2597740		0.2809011	34.879934

Example: Design a 4th-order Chebuchev filter with

- A DC gain of 1.000
- A ripple of 0.2, and
- A corner at 10 rad/sec

Solution:

$$G(s) = \left(\frac{7.2^2 \cdot 11.1^2}{\left(s+7.2\angle\pm38.5^0\right)\left(s+11.1\angle\pm77.8^0\right)} \right)$$

or

$$G(s) = \left(\frac{6387}{\left(s^2+11.27s+51.84\right)\left(s^2+4.69s+123.21\right)} \right)$$

Checking the gain vs. frequency in Matlab

```
p1 = 7.2 * exp(j*38.5*pi/180);
p2 = conj(p1);
p3 = 11.1 * exp(j*77.8*pi/180);
p4 = conj(p3);

w = [0:0.01:30]';
s = j*w;

Gd = 1 * (w < 10) .* (w > -10);

Gs = 7.2^2 * 11.1^2  ./ ( (s+p1) .* (s+p2) .* (s+p3) .* (s+p4) );

plot(w,abs(Gs),'b',w,abs(Gd),'m');
xlabel('Frequency (rad/sec)');
ylabel('Gain');
```
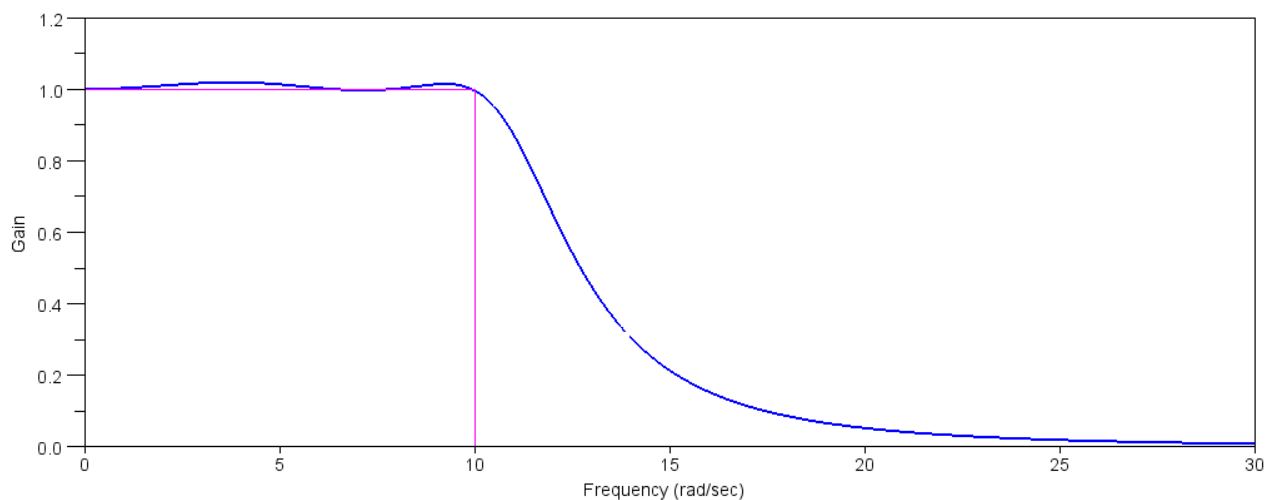

Gain vs. Frequency for a 4th-Order Chebychev Filter with a Corner at 10 rad/sec

Elliptic Filters:

An Elliptic filter adds zeros to force the gain to drop quickly at the corner. The tradeoff is the highfrequency gain doesn't roll off like they do with Butterworth of Type-1 Chebychev filters. An elliptic filter with an attenuation of 60dB in the reject region follows:

Pass	W1 = 0 to 1	W1 = 0 to 1	W1 = 0 to 1	W1 = 0 to 1
Reject	W2 = 9 to infinity	W2 = 3 to infinity	W2 = 1.7 to infinity	W2 = 1.3 to infinity
zeros	j 9.919	j3.246 j7.705	j1.831 j2.907	j1.316 j1.524 j2.491
poles	-0.391 + j1.242 -0.942	-0.572 + j0.467 -0.221 + j1.076	-0.494 -0.365 + j0.660 -0.143 + j1.005	-0.437 -0.331 + j0.583 -0.159 + j0.907 -0.040 + j1.024

Note that zeros are placed on the jw axis. These force the gain to zero at these frequencies - and close to zero near them. With the use of complex zeros, you can force the gain to drop more quickly past the passband. TheThe drawback is the gain 'just' remains below 0.001 rather than rolling off to zero as it would do with all poles and no zeros.
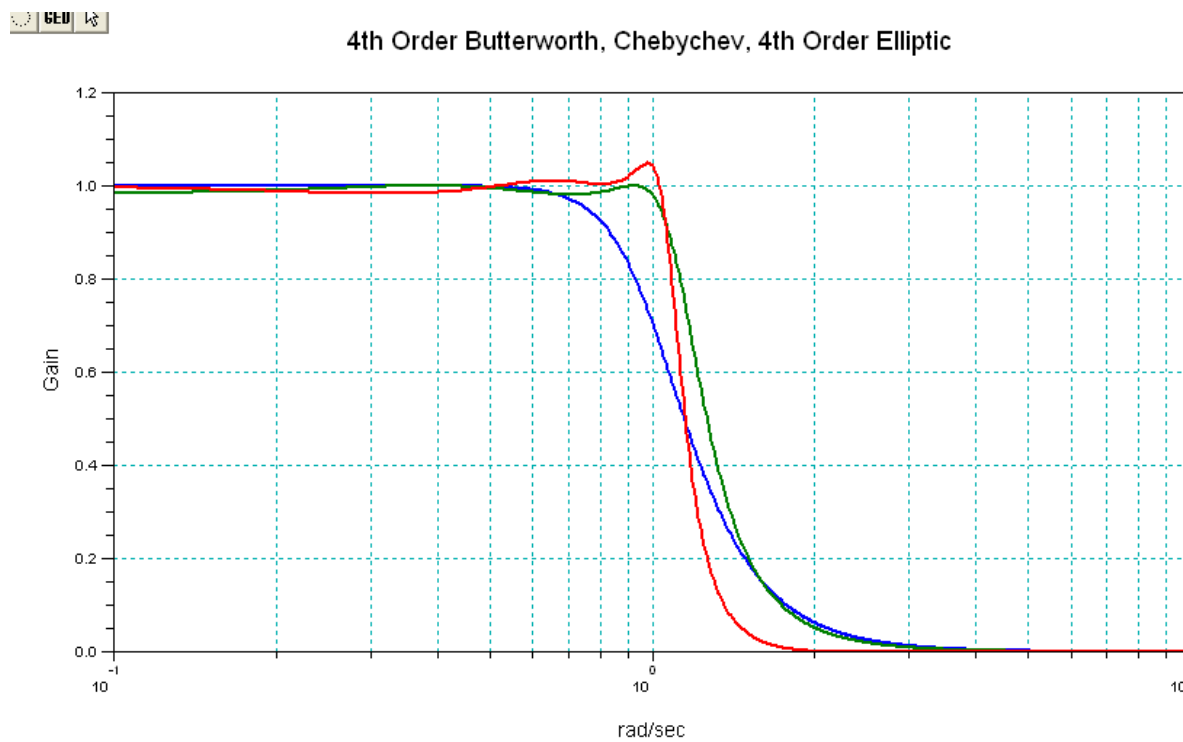
Also note that the closer the reject region is to the passband, the more poles and zeros are required.


Comparison of Gains vs. Frequency:
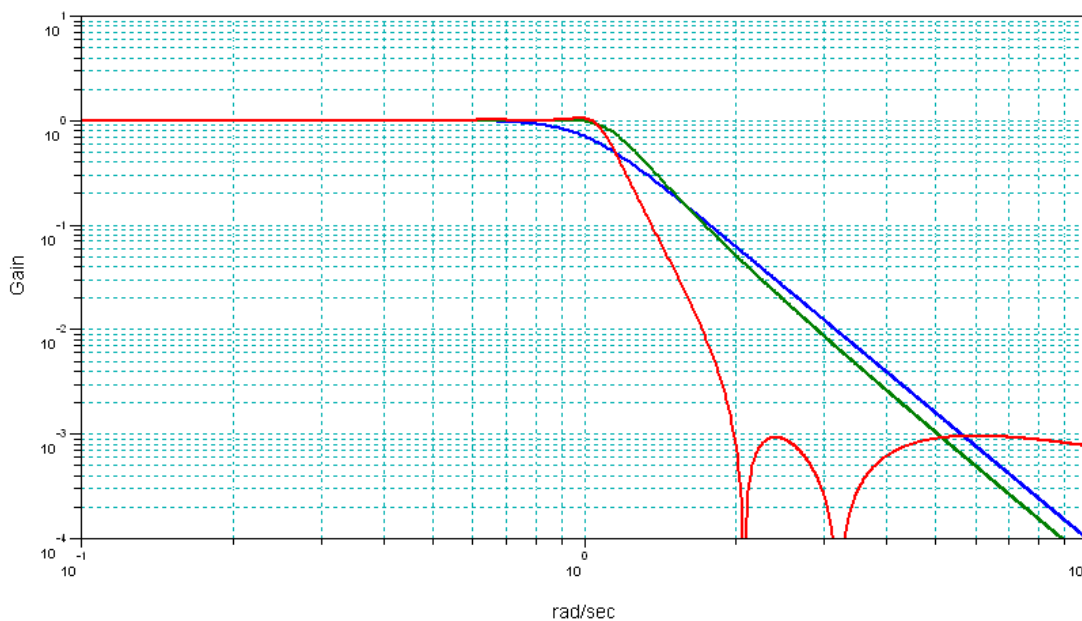
Gain vs. Frequency for the Butterworth filter (blue), Chebychev filter (green), and Elliptic filter (red).

In the above gain vs. frequency, note that the Elliptic filter has a gain which goes to zero at 2.078 and 3.212 rad/sec. In the reject region, the gain is less than 0.001 - which was one of the design constraints in the elliptic filter presented here.

The Chebychev and Butterworth filter both have gains that keep dropping off as ω^{-4} since these filters have four poles and no zeros. If you want the gain to keep rolling off, the Butterworth or Elliptic filters are better. 0.001 is small enough, the Elliptic filter is the closest approximation to an ideal low pass filter.
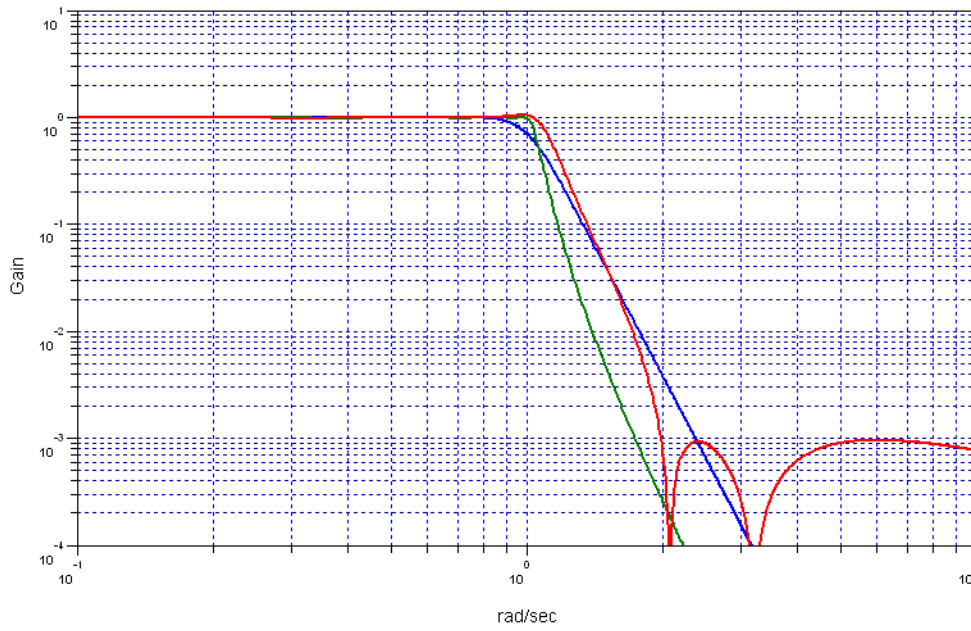
8th-Order Filters
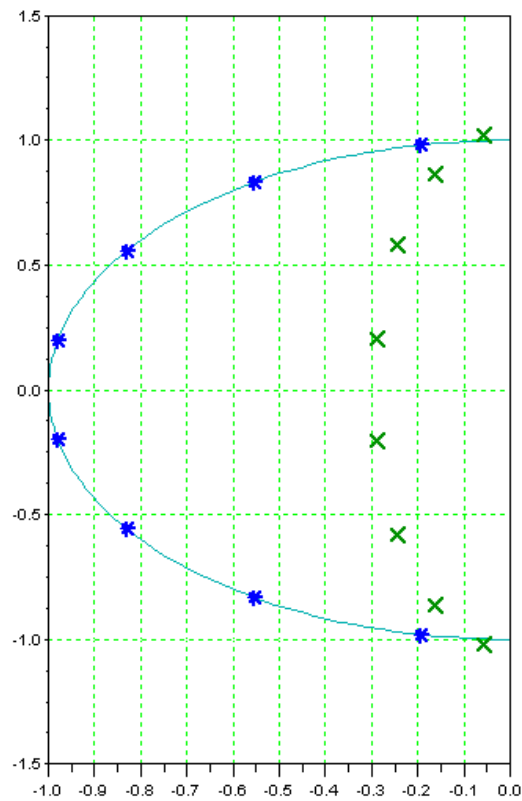
Pole location of a Butterworth filer (blue) and Chebuchef filter (green) along with the unit circle.

The gain vs. frequency is show below. Note that the Chebychev filter trades off ripple in the passband for a faster rolloff outside the passband. They both have a gain which drop off as ω^{-8} with eight poles and no zeros.
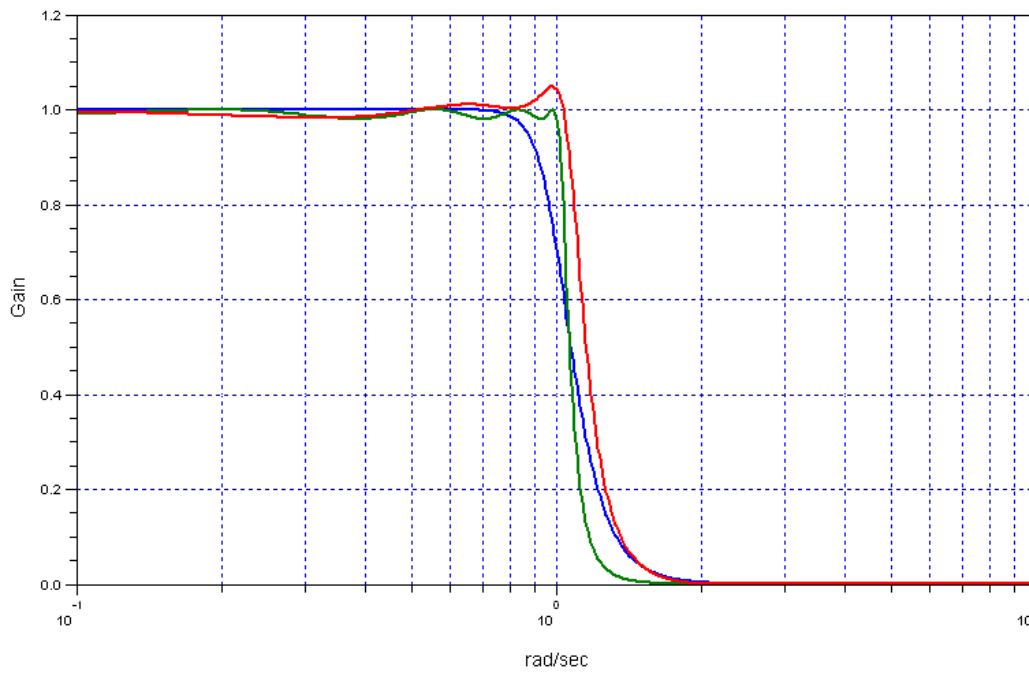
8th Order Butterworth, Chebychev, 4th Order Elliptic

8th Order Butterworth, Chebychev, 4th Order Elliptic

SciLab Code:

zpell - lowpass elliptic filter

```
Calling Sequence

[zeros,poles,gain]=zpell(epsilon,A,omegac,omegar)

Parameters

 epsilon : real : ripple of filter in pass band ( 0<epsilon<1 )
 A : real : attenuation of filter in stop band ( A>1 )
 omegac : real : pass band cut-off frequency in Hertz
 omegar : real : stop band cut-off frequency in Hertz
 zeros : resulting zeros of filter
 poles : resulting poles of filter
 gain : resulting gain of filter

Description

Poles and zeros of prototype lowpass elliptic filter.  gain  is the gain of the
filter
```

zpbutt - Butterworth analog filter

```
Calling Sequence

[pols,gain]=zpbutt(n,omegac)

Parameters

 n : integer (filter order)
 omegac : real (cut-off frequency in Hertz)
 pols : resulting poles of filter
 gain : resulting gain of filter

Description

computes the poles of a Butterworth analog filter of order  n  and cutoff
frequency omegac transfer function H(s) is calculated by
H(s)=gain/real(poly(pols,'s'))
```

zpch1 - Chebyshev analog filter

```
Calling Sequence

[poles,gain]=zpch1(n,epsilon,omegac)

Parameters

 n : integer (filter order)
 epsilon : real : ripple in the pass band ( 0<epsilon<1 )
 omegac : real : cut-off frequency in Hertz
 poles : resulting filter poles
 gain : resulting filter gain

Description
```

Poles of a Type 1 Chebyshev analog filter. The transfer function is given by :

H(s)=gain/poly(poles,'s')