Active Filters

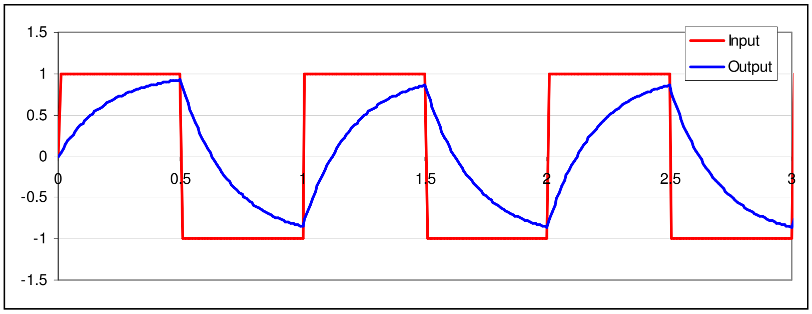
Background:

Filters are circuits whose behaviour changes with frequency. Essentially, all circuits with capacitors and/or inductors are filters.

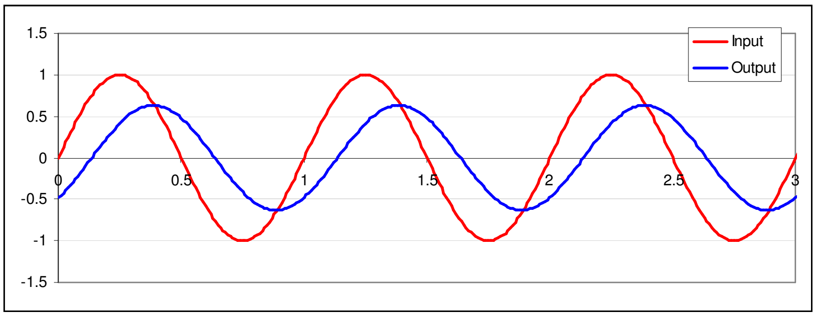
When analyzing a filter, sinusoids are used for the inputs and outputs. Sinusoids are very special signals. For example, assume you apply a 1Hz signal to an RC filter which satisfies the following differential equation

$$\frac{dy}{dt} + 5y = 5x$$

where y is the output of the RC filter and x is the input. If the input is a 1Hz square wave, the output will not be a square wave. The relationship between the two is not a simple one.



If the input is a 1Hz sine wave, however, the output is also a 1Hz sine wave.



A simple way to relate the two signals is the change in amplitude and the phase shift.

Phasor analysis is a quick way to determine the output of a filter (or the solution to a differential equation) for sinusoidal inputs. Since you know the output will also be a sine wave of the same frequency as the input, all you need to know is the amplitude and the phase shift of the output.

For example, assume you want to find the solution to the above differential equation for

$$x(t) = \sin(2\pi t)$$

Using phasor analysis, replace all derivatives with 's' and solve for the output, Y:

$$sY + 5Y = 5X$$

$$Y = \left(\frac{5}{s+5}\right)X$$

The function $\left(\frac{5}{s+5}\right)$ is the gain from the input to the output at all frequencies. Since you only care about the gain at 1Hz, analyze this as

$$\left(\frac{5}{s+5}\right)_{s=j2\pi} = 0.627\angle -51.5^0$$

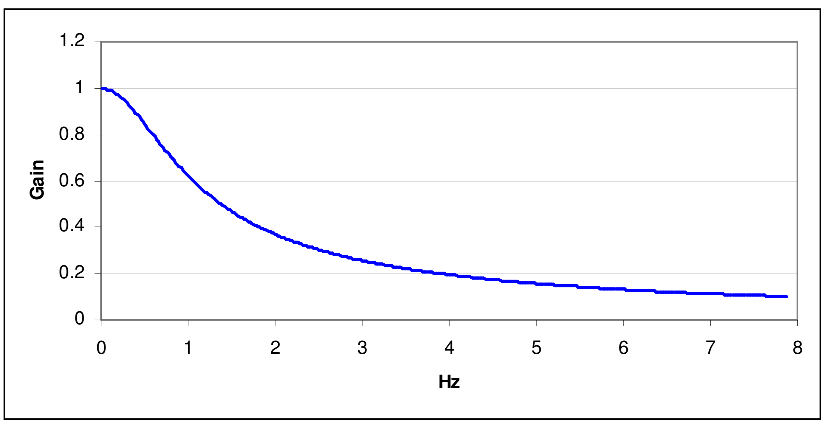
This means that at 1Hz, the output will be 0.627 of the input, shifted -51.5 degrees. If

$$x(t) = \sin(2\pi t)$$

then

$$y(t) = 0.627\sin(2\pi t - 51.5^0)$$

Note from (4) that the gain from the input to the output is a function of frequency. If you plot the magnitude of the gain vs. frequency, you can get a feeling for how this system behaves:



When the input is less than 0.5Hz, the gain is almost one: the output is almost as large as the input. When the input is past 2Hz (or so), the gain drops. In short, this filter passes low frequencies (near zero) and rejects high frequencies. Hence, it is called a low-pass filter.

Active Filters

A filter is simply a circuit where the gain is a function of frequency. An active filter is a filter which includes one or more op-amps. What including an op-amp does for you is it allows:

- Gains larger than one
- High input impedances, reducing the loading effect of the filter on the driving circuit
- Low output impedances, allowing the filter to drive various items, and
- A filter with real and/or complex poles without using inductors.

Inductors tend to be large, lossy, prone to coupling, and expensive. Circuits which only use capacitors and resistors tend to work much better.

In general, a filter will be of the form

$$G(s) = k\left(\frac{(s+z_1)(s+z_2)}{(s+p_1)(s+p_2)(s+p_3)}\right)$$

where

zi are the zeros of the filter,

pi are the poles of the filter, and

k is a gain.

Today's lecture covers different circuits to implement a filter with

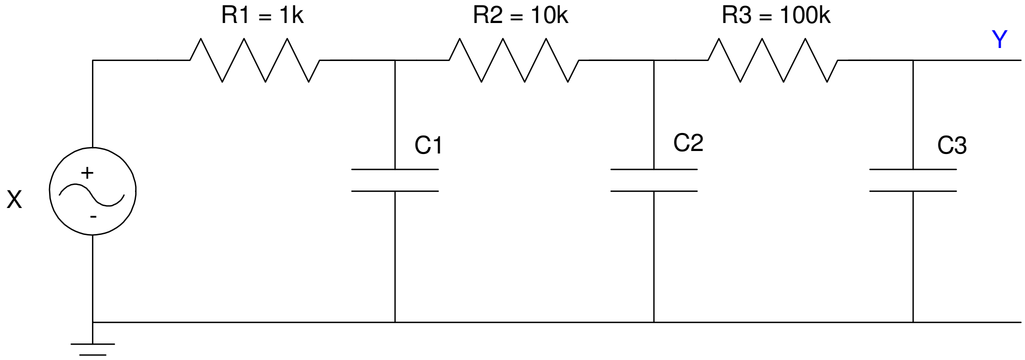
- Real poles, and
- Complex Poles

Real Poles: Passive RC Filters

Problem: Design a circuit to implement

$$Y = \left(\frac{abc}{(s+a)(s+b)(s+c)}\right)X$$

Solution: One solution is to use an RC filter:



The transfer function is then

$$Y \approx \left(\frac{abc}{(s+a)(s+b)(s+c)}\right)X$$

where

$$a = \left(\frac{1}{R_1C_1}\right)$$

$$b = \left(\frac{1}{R_2C_2}\right)$$

$$c = \left(\frac{1}{R_3C_3}\right)$$

Notes:

- This filter is easy to build (good), but
- It's not a very good filter (gain drops off with frequency very fast)

Also, the reason the resistor goes up 10x each state is to prevent loading. If R2 did not exist, then gain to V1 would be

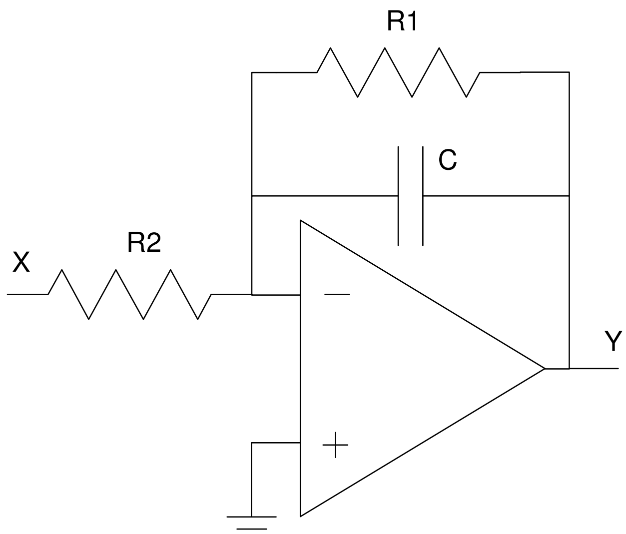
$$V_1 = \left(\frac{a}{s+a}\right)X$$

where

$$a = \left(\frac{1}{R_1C_1}\right)$$

By adding R2, current is shunted from C1, changing the circuit. By making R2 large relative to R1, this loading is there but is small.

Real Poles, No Zeros (take 2)



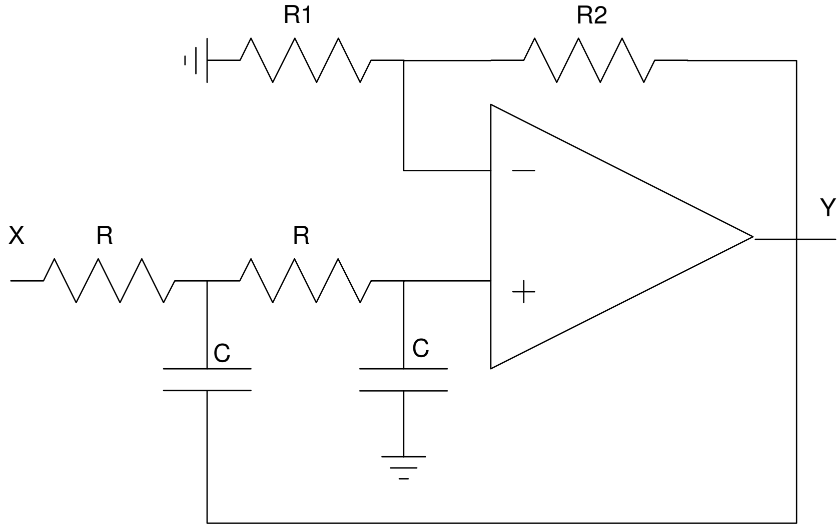
$$Y = -\left(\frac{a}{s+b}\right)X$$

where

$$a = \frac{1}{R_2 C}$$

$$b = \frac{1}{R_1 C}$$

Complex Poles, No Zeros



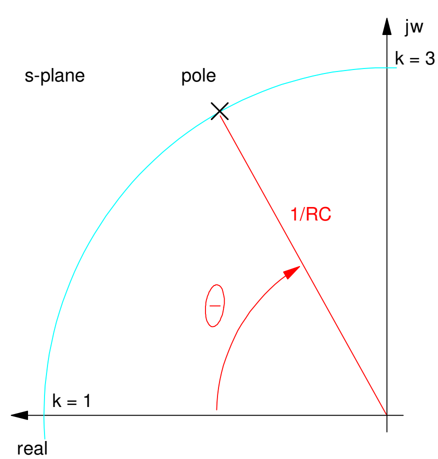
$$Y = \left(\frac{k \cdot \left(\frac{1}{RC}\right)^2}{s^2 + \left(\frac{3-k}{RC}\right)s + \left(\frac{1}{RC}\right)^2} \right) X$$

This filter has two complex poles with

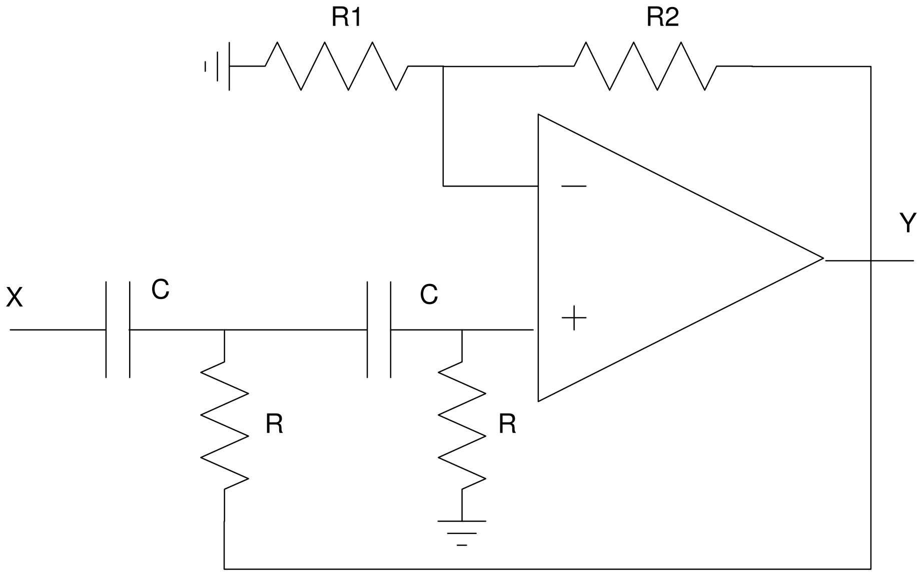
- Amplitude = $\frac{1}{RC}$
- Angle: $3 - k = 2\cos\theta$
- DC gain $k = \left(1 + \frac{R_2}{R_1}\right)$

Note that the angle of the poles goes from

- 0 degrees when k =1
- 90 degrees when k = 3 (an oscillator)



Comples Polex, Two Zeros at s = 0

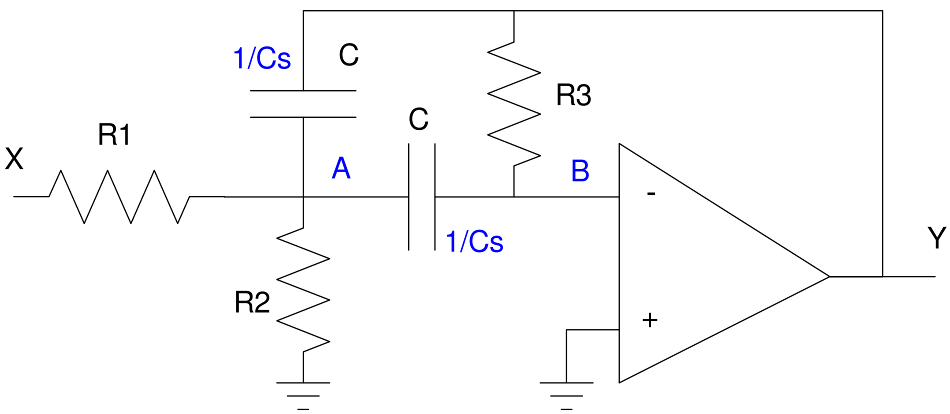


$$Y = \left(\frac{k \cdot s^2}{s^2 + \left(\frac{3-k}{RC}\right) s + \left(\frac{1}{RC}\right)^2} \right) X$$

This filter has two complex poles with

- Amplitude = $\frac{1}{RC}$
- Angle: $3 - k = 2 \cos \theta$
- High Freq gain $k = \left(1 + \frac{R_2}{R_1}\right)$

Comples Polex, One Zeros at s = 0: $Y = \left(\frac{as}{s^2+bs+c}\right)X$



$$Y = \left(\frac{-\left(\frac{1}{R_1 C}\right)s}{s^2+\left(\frac{2}{R_3 C}\right)s+\left(\frac{R_1+R_2}{R_1 R_2}\right)\left(\frac{1}{R_3 C^2}\right)}\right)X$$

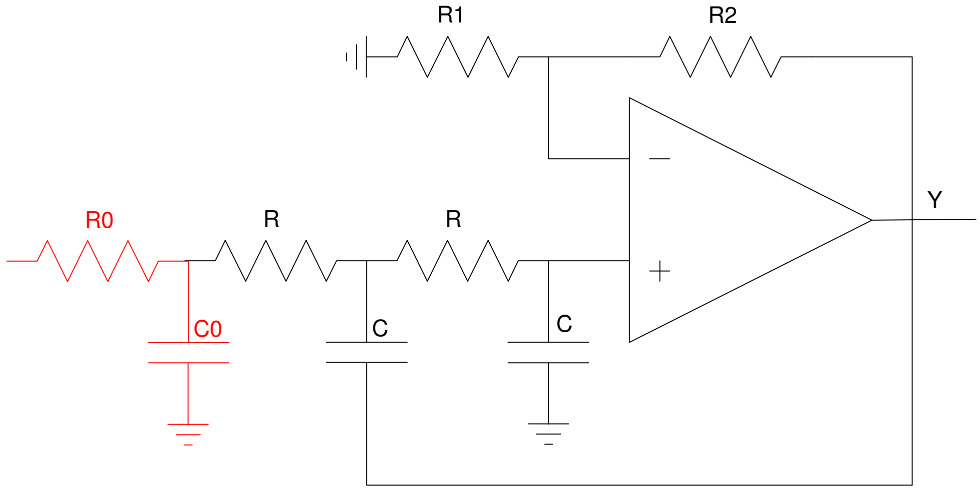
Example: Design a circuit to implement

$$Y = \left(\frac{1,244,485}{(s+85)(s+121\angle 69.5^0)(s+121\angle -69.5^0)} \right) X$$

Rewrite this as

$$Y = \left(\frac{85}{s+85} \right) \left(\frac{14,641}{(s+121\angle 69.5^0)(s+121\angle -69.5^0)} \right) X$$

Use the previous filters



$$\left(\frac{\left(\frac{1}{R_0 C_0}\right)}{s+\left(\frac{1}{R_0 C_0}\right)} \right) \left(\frac{k \cdot \left(\frac{1}{RC}\right)^2}{s^2+\left(\frac{3-k}{RC}\right)s+\left(\frac{1}{RC}\right)^2} \right)$$

To avoid loading, let

- R0 = 10k
- R = 100k

Matching terms in the denominator:

$$\left(\frac{1}{R_0 C_0}\right) = 85 \qquad C_0 = 1.17\mu F$$

$$\left(\frac{1}{RC}\right) = 121 \qquad C = 0.082\mu F$$

$$3 - k = 2\cos(69.5^0)$$

$$k = 2.3$$

$$1 + \frac{R_2}{R_1} = 2.3$$

R1 = 100k, R2 = 1.3k

Note: This circuit has a DC gain of 2.3 (instad of 1.0). Just note this and call the output 2.3Y.

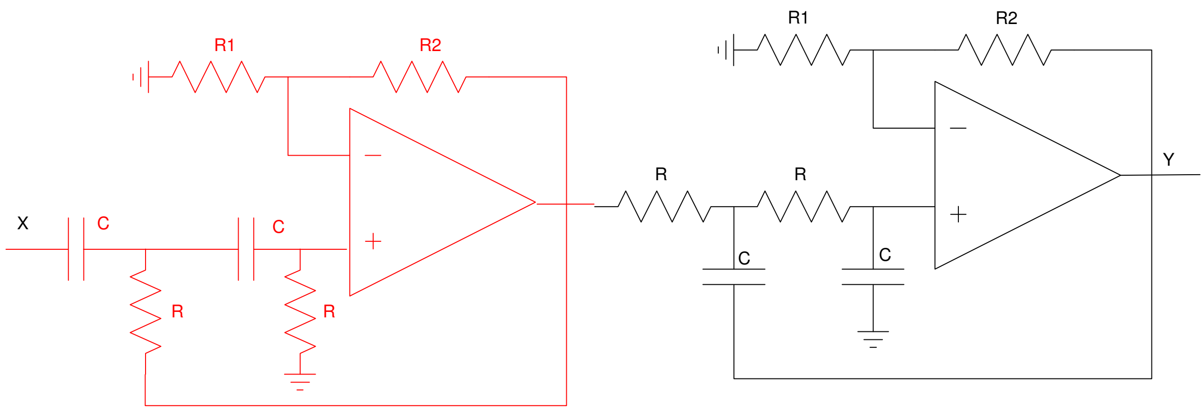
Example: Design a filter to implement

$$Y = \left(\frac{100{,}000 s^2}{\left(s^2+14s+100\right)\left(s^2+100s+10{,}000\right)} \right) X$$

Solution: Rewrite this as the product of two filters:

$$Y = \left(\frac{s^2}{\left(s^2+14s+100\right)} \right) \left(\frac{10{,}000}{\left(s^2+100s+10{,}000\right)} \right) X$$

Using the previous circuits (building blocks),



$$\left(\frac{k \cdot s^2}{s^2 + \left(\frac{3-k}{RC}\right)s + \left(\frac{1}{RC}\right)^2} \right) \qquad \left(\frac{k \cdot \left(\frac{1}{RC}\right)^2}{s^2 + \left(\frac{3-k}{RC}\right)s + \left(\frac{1}{RC}\right)^2} \right)$$

Matching the poles:

$$\left(\frac{1}{RC}\right)^2 = 100 \qquad \left(\frac{1}{RC}\right)^2 = 10,000$$

R = 100k | R = 100k

C = 0.1uF | C = 0.001uF

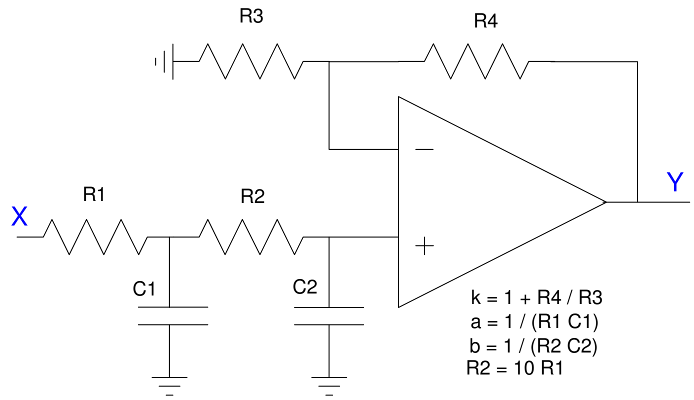
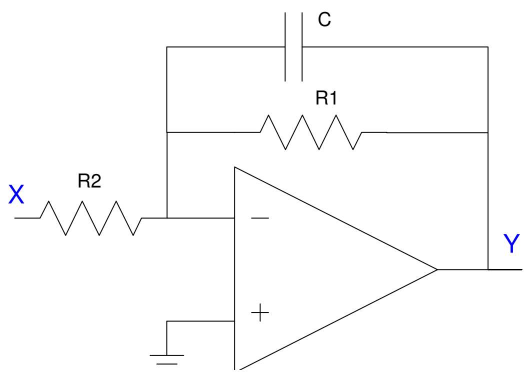
$$3 - k = 2\cos\left(45^0\right) \qquad 3 - k = 2\cos\left(60^0\right)$$

k = 1.5858 | k = 2

R1 = 100k | R1 = 100k

R2 = 58k | R2 = 100k

Note: This circuit has a DC gain of 3.17 (instad of 1.0). Just note this and call the output 3.17Y.

Filter Description	G(s)	Circuit
Real Poles No Zeros DC gain > 1	$\left(\frac{kab}{(s+a)(s+b)}\right)$; $a = \frac{1}{R_1C_1}$ $b = \frac{1}{R_2C_2}$ $R_2 = 10R_1$ $k = 1 + \frac{R_4}{R_3}$	
Single Real Pole No Zeros	$\left(\frac{-a}{s+b}\right)$; $b = \frac{1}{R_1C}$ $\frac{a}{b} = \frac{R_1}{R_2}$	
Complex Poles No Zeros	$\left(\frac{ka^2}{(s+a\angle\theta)(s+a\angle-\theta)}\right)$; $a = \frac{1}{RC}$ $k = 1 + \frac{R_4}{R_3}$ $3 - k = 2\cos\theta$